Integers, without large prime factors, in arithmetic progressions, I

by

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Abstract: We give a variety of estimates for the number of integers, free of large prime factors, in arithmetic progressions. In particular we show that we get approximately the same number of integers up to x that are composed only of primes $\leq y$ in each arithmetic progression (mod q), provided that $q \leq y^{\varepsilon}$ for some small fixed $\varepsilon > 0$; this extends the ranges for x and q of previous estimates.

1. Introduction.

Let $\Psi(x,y)$ be the number of integers $\leq x$ that are free of prime factors $> y$, $\Psi_q(x,y)$ be the number of such integers that are also coprime to q, and $\Psi(x,y;a,q)$ be the number of such integers in the congruence class a (mod q). Good estimates for these functions have many applications in number theory (for instance, to bounds for the least quadratic non–residue (mod$p$) [Bg], to Waring's problem [Va], to finding large gaps between primes [Ra] and to analysis of factoring algorithms [Po]), as well as being interesting in of themselves, and so have been extensively investigated.

Recently Hildebrand and Tenenbaum [HT] have provided a good estimate for $\Psi(x,y)$ for all $x \geq y \geq 2$, and a similar method works for $\Psi_q(x,y)$; however their method applies to $\Psi(x,y;a,q)$ only when q is considerably smaller than y. In general one expects that, for sufficiently large x,

$$\Psi(x,y;a,q) \sim \frac{\Psi_q(x,y)}{\phi(q)} \tag{1.1}$$

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whenever a is coprime to q, provided that the primes $\leq y$ generate the full multiplicative group of units (mod$q$). Buchstab [Bu] proved such a result when q and $u(:= \log x/\log y)$ are fixed; and extensions of this, for q up to a fixed power of $\log y$, are considered in [LF] and [No]. Recently Fouvry and Tenenbaum [FT] have shown that the estimate

$$\Psi(x,y;a,q) = \frac{\Psi_q(x,y)}{\phi(q)}\left\{1+O\Big(\exp(-c\sqrt{\log y})\Big)\right\} \tag{1.2}$$

holds uniformly for

$$y \geq 2, \qquad x \geq y \geq \exp(c(\log\log x)^2), \tag{1.3}$$

and all q up to any fixed power of $\log x$; they also gave a similar estimate (which implies (1.1)) for all $q \leq \exp(c\sqrt{\log y})$. As one might guess from these ranges for q, the methods used to obtain these estimates depend on an understanding of the distribution of zeros of Dirichlet L-functions (mod q). Here we avoid this difficulty and so increase the ranges in which (1.1) is known to hold:

Theorem 1. *Fix $N > 0$. The estimate*

$$\Psi(x,y;a,q) = \frac{\Psi_q(x,y)}{\phi(q)}\left\{1+O\left(\frac{\log q}{\log y}\right)\right\} \tag{1.4}$$

holds uniformly in the range

$$(a,q)=1, \qquad x \geq y \geq 2, \qquad q \leq \min\{x, y^N\}. \tag{1.5}$$

Note that (1.5) only provides a non–trivial lower bound for $\Psi(x,y;a,q)$ if $q < y^\varepsilon$ for some sufficiently small $\varepsilon > 0$.

Fouvry and Tenenbaum also showed that (1.1) holds, in a certain 'average' sense, in a much wider range than (1.5); specifically, that there exists a value of $B > 0$, such that if $\log y \gg \log x \log\log\log x/\log\log x$ then (1.1) holds for almost all $q \leq x^{1/2}/\log^B x$ for all a coprime to q. We use their theorem to obtain a result for the same values of q and y, but now for *all* sufficiently large x (independent of y):

Theorem 2. *For each $A > 0$ there exist constants $B = B(A) > 0$ and $C = C(A) > 0$, such that the estimate*

$$\sum_{q\le Q} \max_{x'\le x}\max_{(a,q)=1}\left|\Psi(x',y;a,q)-\frac{\Psi_q(x',y)}{\phi(q)}\right| \ll_A \frac{\Psi(x,y)}{\log^A y} \tag{1.6}$$

holds uniformly for

$$y\ge 100,\quad 1\le Q\le \exp(C\log y\,\log\log y/\log\log\log y),\quad x\ge Q^2\log^B Q. \tag{1.7}$$

Theorem 1 provides a non–trivial lower bound for $\Psi(x,y;a,q)$ only when q is less than a sufficiently small power of y. By using a recent result of Balog and Pomerance we can extend the range for q at the expense of a weaker error term:

Theorem 3. *Fix N in the range $0 < N < 4/3$ and $\varepsilon > 0$. The estimate*

$$\Psi(x,y;a,q)\asymp \frac{1}{\phi(q)}\Psi_q(x,y) \tag{1.8}$$

holds uniformly in the range

$$(a,q)=1,\ y\ge 2,\ q\le y^N,\ x\ge \max\{y^{3/2+\varepsilon}, yq^{3/4+\varepsilon}\}. \tag{1.9}$$

We may also obtain a strong upper bound for $\Psi(x,y;a,q)$ in a much wider range than in Theorem 1, which can be used to improve a result of Friedlander [Fr].

Theorem 4. *There exists a constant $c > 0$ such that the inequality*

$$\Psi(x,y;a,q)\le c\frac{\Psi_q(x,y)}{\phi(q)}\Bigg/\min_{q\le x'\le q\max\{y,\log^2 q\}}\left(\frac{\Psi_q(x',y)}{\frac{\phi(q)}{q}x'}\right) \tag{1.10}$$

holds for all positive integers q and all $x\ge q$.

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Notation: Throughout c and ε are taken to be absolute positive constants; however, they may change value from one proof to another. If d is coprime with q then we define $\pi(x; q, a/d)$ (or $\Psi(x, y; a/d, q)$) to be the number of primes $\leq x$ (or the number of integers $\leq x$ that are free of prime factors $> y$), which belong to the arithmetic progression $a/d \pmod q$.

2. The key idea.

The function $\Psi(x, y)$ has been extensively investigated (see [No] for a review up to 1971): In 1930, Dickman [Di] showed that for any fixed $u > 0$,

$$\Psi(x, y) \sim x\rho(u) \qquad (x \to \infty, \quad y = x^{1/u}) \tag{2.1}$$

where $\rho(u)$, the *Dickman function*, equals 1 for $0 \leq u \leq 1$, and is the continuous solution of the differential difference equation $u\rho'(u) + \rho(u-1) = 0$ for $u > 1$.

In 1951 de Bruijn [dB], by carefully treating Buchstab's equation

$$\Psi(x, y) \ = \ \Psi(x, z) + \sum_{\substack{z < p \leq y \\ p \text{ prime}}} \Psi\left(\frac{x}{p}, p\right)$$

as an approximate functional equation for $x\rho(u)$ (that is with $\Psi(x, y)$ replaced by $x\rho(u)$), gave the estimate

$$\Psi(x, y) = x\rho(u) \left\{ 1 + O_\varepsilon\left(\frac{\log (u+1)}{\log y} \right) \right\} \tag{2.2}$$

for $u = \log x / \log y$, when $x \geq y \geq \exp((\log x)^{5/8+\varepsilon})$, for any fixed $\varepsilon > 0$.

Recently Hildebrand [H1] considered instead the equation

$$\Psi(x,y)\log x = \int_1^x \frac{\Psi(t,y)}{t}\mathrm{d}t + \sum_{\substack{p^m\le x\\ p\le y}} \Psi\left(\frac{x}{p^m},y\right)\log p \tag{2.3}$$

as an approximate functional equation for $x\rho(u)$, and proved (2.2) in the much wider range

$$x\ge 2, \qquad x\ge y\ge \exp\left(c(\log\log x)^{5/3+\varepsilon}\right), \tag{2.4}$$

for any fixed $\varepsilon>0$.

Any such method has the limitation that the approximation involved (that is, of the functional equation for $\Psi(x,y)$ serving as an approximate functional equation for $x\rho(u)$) gets worse as u gets larger, and so limits the range of u for which (2.1) is provable. In fact, both De Bruijn's and Hildebrand's results depend on the size of the error term in the Prime Number Theorem (to determine the accuracy of the approximation); and Hildebrand has even shown ([H2]) that (2.3) holds for all $y\ge \log^{2+\varepsilon} x$ if and only if the Riemann Hypothesis is true. However we do not believe that (2.1) can hold uniformly for $y=\log^{2-\varepsilon} x$, for any fixed $\varepsilon>0$.

The main new idea in this paper is that we establish a functional identity for $\Psi(x,y;a,q)$, which remains valid when we replace each term of the form $\Psi(t,y;b,q)$ with $\frac{1}{\phi(q)}\Psi_q(t,y)$. This has the benefit that if we consider this functional identity for $\Psi(x,y;a,q)$ as an approximate functional equation for $\frac{1}{\phi(q)}\Psi_q(x,y)$, then we lose nothing in this approximation. In Proposition 1 below we show that this means that we can reduce the question of establishing (1.1) for all $x\ge x_0$, to that of establishing (1.1) in an interval of the form $x_0\le x\le x_1$.

Thanks to a suggestion by the anonymous referee, our proof of this is entirely elementary, and so we shall prove a generalization of the result indicated above:

Suppose that P is a set of primes, each $\le y$, none of which divide the positive integer q. Define $\Psi(x,P)$ to be the number of integers $\le x$ whose prime factors all belong to the set P, and $\Psi(x,P;a,q)$ to be the number of such integers that are congruent to $a \pmod q$.

Let $G=G(q,P)$ be the multiplicative subgroup of $(\mathbf{Z}/q\mathbf{Z})^*$ generated by the primes in P; note that $\Psi(x,P;a,q)\neq 0$ for some x if and only if $a\in G$. In general one expects that, for sufficiently large x,

$$\Psi(x,P;a,q)\sim\frac{\Psi(x,P)}{|G|}\qquad\text{for all } a\in G \tag{2.5}$$

We shall prove the following result:

Proposition 1. *There exists a constant $c>0$ such that for any non–empty set of primes P, each $\leq y$, none of which divide the positive integer q, and any $\lambda\geq 0$, $x_0=y^{u_0}$, $x_1=y^{u_1}\geq 2x_0\geq 4$, with $\overline{u}_1=\min\{u_1,|P|\}$, we have*

$$\begin{aligned}\left|\frac{\Psi(x,P;a,q)}{\Psi(x,P)}-\lambda\right| \leq \max_{\substack{x_0\leq x'\leq x_1\\ b\in G}}\left|\frac{\Psi(x',P;b,q)}{\Psi(x',P)}-\lambda\right| &+\\ +\ c\,\frac{|P|(u_0+1)\log y}{\overline{u}_1\Psi(x_1,P)}&\left\{\max_{b\in G}\ \Psi(x_0,P;b,q)+\lambda\Psi(x_0,P)\right\}\end{aligned} \tag{2.6}$$

for all $x\geq x_0$ and $a\in G=G(q,P)$.

Typically, we apply Proposition 1 with λ equal to its 'expected value', $1/|G|$. If $u_0\leq|P|$ then, in order to deduce (2.5) for all $x\geq x_0$, $a\in G$ from Proposition 1, we must be able to choose x_1 so that $\Psi(x_0,P)\ |P|\log y=o(\Psi(x_1,P))$, and also so that (2.5) holds for all $x_0\leq x\leq x_1$, $a\in G$. To deduce (1.1) we just consider the case where P is the set of all primes $\leq y$ that do not divide q. This essentially describes the main ideas behind our results.

3. The Proof of Proposition 1.

We start by giving functional equations for $\Psi(x,P;a,q)$ and $\Psi(x,P)$, analogous to Hildebrand's equation (2.3): The idea is to evaluate $\sum_{\substack{n\leq x,\ n\equiv a \pmod q\\ p|n\Rightarrow p\in P}}\log n$ in two different ways. First by partial summation, and second by writing each $\log n$ as $\sum_{p^m|n} \log p$, and then swapping the order of summation. This leads to the identity

$$\Psi(x,P;a,q)\log x \;=\; \int_1^x \frac{\Psi(t,P;a,q)}{t}\mathrm{d}t + \sum_{p\in P, p^m\le x} \Psi\Big(\frac{x}{p^m},P;\frac{a}{p^m},q\Big)\log p\ . \tag{3.1}$$

Summing (3.1) over all integers $a \in G$ we get

$$\Psi(x,P)\log x \;=\; \int_1^x \frac{\Psi(t,P)}{t}\mathrm{d}t + \sum_{p\in P, p^m\le x} \Psi\Big(\frac{x}{p^m},P\Big)\log p\ . \tag{3.2}$$

Notice that (3.2) is just (3.1) with each term of the form $\Psi(t,P;b,q)$ replaced by $\Psi(t,P)$.

Lemma 1. *For q and P as above, and any real $k \ge 1$, we have*

$$\Psi(xy^k,P) \;\ge\; \frac{1}{2}\frac{k}{u+k}|P|\Psi(x,P).$$

Proof: Replacing x by xy^k in (3.2) we obtain

$$\Psi(xy^k,P)\log(xy^k) \;\ge\; \sum_{p\in P, p^m\le y^k} \Psi\Big(\frac{xy^k}{p^m},P\Big)\log p \;\ge\; \Psi(x,P)\sum_{p\in P}\log p \sum_{m:\ p^m\le y^k} 1,$$

which gives the result as $\sum_{m:\ p^m\le y^k} 1 \ge \frac{1}{2}k\log y/\log p$.

Lemma 2. *For q and P as above, and any positive real u, we have*

$$\sum_{i\ge 0}\frac{1}{(u+i)\Psi(y^{u+i},P)} \;\ll\; \frac{1}{\overline{u}\Psi(y^u,P)}.$$

Proof: We begin by proving this when $1 \le |P| \le 4$. First note that the result for $0 < u \le 1$ comes from the result for $u \ge 1$ by missing out the first term in the sum and noting that $\overline{u} \le 4$. Now, it is easy to establish that $\Psi(x,P) \asymp \prod_{p\in P}\frac{\log x}{\log p}$ holds uniformly for $x \ge y$. Therefore

$$\sum_{i\ge 0}\frac{1}{(u+i)\Psi(y^{u+i},P)} \;\asymp\; \left(\prod_{p\in P}\frac{\log p}{\log y}\right)\sum_{i\ge 0}\frac{1}{(u+i)^{|P|+1}} \;\asymp\; \prod_{p\in P}\frac{\log p}{\log x} \;\asymp\; \frac{1}{\Psi(x,P)},$$

for $u \geq 1$, and the result follows as $\overline{u} \leq 4$.

So now assume that $|P| \geq 5$. Let $u_0 = u$ and for each $j \geq 0$, let k_j be the smallest integer $\geq 20u_j/|P|$ and $u_{j+1} = u_j + k_j$. Lemma 1 implies that $\Psi(y^{u_{j+1}}, P) \geq 2\Psi(y^{u_j}, P)$, and so $\Psi(y^{u_j}, P) \geq 2^j \Psi(y^u, P)$ for all $j \geq 0$. Thus

$$\sum_{i\geq 0} \frac{1}{(u+i)\Psi(y^{u+i}, P)} \leq \sum_{j\geq 0} \frac{u_{j+1} - u_j}{u_j \Psi(y^{u_j}, P)} \leq \frac{1}{\Psi(y^u, P)} \sum_{j\geq 0} \frac{k_j}{u_j 2^j},$$

and the result follows as each $k_j/u_j \ll 1/u_j + 1/|P| \ll 1/\overline{u}$.

The Proof of Proposition 1: The general form of this argument owes much to the proofs in [H1]. Define, for each $x \geq 1$,

$$\Gamma(x) = \Gamma_q(x) := \max_{a\in G} \left| \frac{\Psi(x, P; a, q)}{\Psi(x, P)} - \lambda \right|, \quad \text{and} \quad \Gamma^*(x) = \Gamma_q^*(x) = \max_{x_0 \leq x' \leq x} \Gamma(x')$$

for each $x \geq x_0$. By subtracting λ times (3.2) from (3.1), where $a \in G$ is selected so that $|\Psi(x, P; a, q) - \lambda\Psi(x, P)|$ is maximal, we have, for $x \geq 2x_0$,

$$\Gamma(x)\Psi(x, P)\log x \leq \int_{x_0}^{x} \Gamma(t)\frac{\Psi(t, P)}{t} dt + \sum_{p\in P, p^m \leq x/x_0} \Gamma\left(\frac{x}{p^m}\right) \Psi\left(\frac{x}{p^m}, P\right) \log p + \\ + H_q(x; P),$$

where $H_q(x, P)$ equals the contribution of those terms $\Psi(t, P; a, q)$ with $t \leq x_0$ in (3.1) plus λ times the contribution of those terms $\Psi(t, P)$ with $t \leq x_0$ in (3.2). Now, using the trivial inequalities

$$\Psi(t, P; a, q) \leq \Psi(x_0, P; a, q) \quad \text{and} \quad \Psi(t, P) \leq \Psi(x_0, P) \quad \text{for all } t \leq x_0,$$

we find that

$$\begin{aligned} H_q(x; P) &\leq \left\{ \max_{b\in G} \Psi(x_0, P; b, q) + \lambda\Psi(x_0, P) \right\} \left\{ \int_1^{x_0} \frac{dt}{t} + \sum_{p\in P,\ x/x_0 \leq p^m \leq x} \log p \right\} \\ &\leq \left\{ \max_{b\in G} \Psi(x_0, P; b, q) + \lambda\Psi(x_0, P) \right\} (|P| + 1)\log(x_0 y), \end{aligned}$$

a constant, independent of x, which we will henceforth denote by h_q. Inserting this estimate into the inequality above, we obtain, for $x \geq 2x_0$,

$$\Gamma(x)\Psi(x,P)\log x \;\leq\; \Gamma^*(x/2)\left\{\int_{x_0}^{x}\frac{\Psi(t,P)}{t}\mathrm{d}t \;+\; \sum_{p\in P,\;p^m\leq x/x_0}\Psi\left(\frac{x}{p^m},P\right)\;\log p\right\}\;+$$

$$+\;(\Gamma^*(x)-\Gamma^*(x/2))\;\int_{x/2}^{x}\frac{\Psi(t,P)}{t}\mathrm{d}t\;+\;h_q$$

$$\leq \Gamma^*(x/2)\Psi(x,P)\log x \;+\; (\Gamma^*(x)-\Gamma^*(x/2))\Psi(x,P)\log 2\;+\;h_q \tag{3.3}$$

by (3.2) and the trivial inequality

$$\int_{x/2}^{x}\frac{\Psi(t,P)}{t}\mathrm{d}t \;\leq\; \Psi(x,P)\int_{x/2}^{x}\frac{\mathrm{d}t}{t} \;=\; \Psi(x,P)\log 2.$$

For $x \geq 4$ we get, by dividing (3.3) through by $\Psi(x,P)\log x$,

$$\Gamma(x) \;\leq\; \Gamma^*(x/2) \;+\; (\Gamma^*(x)-\Gamma^*(x/2))\frac{\log 2}{\log x} \;+\; \frac{h_q}{\Psi(x,P)\log x}$$

$$\leq\; \frac{1}{2}\Gamma^*(x/2) \;+\; \frac{1}{2}\Gamma^*(x) \;+\; \frac{h_q}{\Psi(x,P)\log x}, \tag{3.4}$$

as $\frac{\log 2}{\log x} \leq \frac{1}{2}$.

Now assume that $x \geq 4x_0$. For any x' in the range $x_0 \leq x' \leq x/2$ we have

$$\Gamma(x') \;\leq\; \Gamma^*(x/2) \;\leq\; \frac{1}{2}\Gamma^*(x/2) \;+\; \frac{1}{2}\Gamma^*(x),$$

as $\Gamma^*(t)$ is a non–decreasing function of t. For any x' in the range $x/2 \leq x' \leq x$ we use (3.4) to obtain

$$\Gamma(x') \;\leq\; \frac{1}{2}\Gamma^*(x'/2) \;+\; \frac{1}{2}\Gamma^*(x') \;+\; \frac{h_q}{\Psi(x',P)\log x'}$$

$$\leq\; \frac{1}{2}\Gamma^*(x/2) \;+\; \frac{1}{2}\Gamma^*(x) \;+\; \frac{h_q}{\Psi(x/2,P)\log (x/2)}\;,$$

as $\Gamma^*(t)$ and $\Psi(t,P)$ are both non–decreasing functions of t. Combining these last two equations, we get

$$\Gamma^*(x) \;\leq\; \Gamma^*(x/2) \;+\; \frac{2h_q}{\Psi(x/2,P)\log (x/2)}\;.$$

Adding together this equation for $x = 2x_1, 4x_1, 8x_1, \dots$ we obtain

$$\Gamma(x) \ \le\ \Gamma^*(x_1) \ +\ 2h_q \sum_{j\ge 0} \frac{1}{\Psi(x_1 2^j, P)\log(x_1 2^j)} \ \le\ \Gamma^*(x_1) \ +\ O\left(\frac{h_q}{\overline{u}_1 \Psi(x_1, P)}\right),$$

for all $x \ge x_0$, by Lemma 2, which is (2.6).

4. Estimates for small x — The Proofs of Theorems 1, 3 and 4.

In [Sa], Saias gave an asymptotic series for $\Psi(x,y)$ in the range (2.4); and the analogous result for $\Psi_q(x,y)$ was proved in [FT], for a wide range of q. An easy consequence of their result is

Lemma 3. *The estimate*

$$\Psi_q(x,y) \ \sim\ \frac{\phi(q)}{q}\Psi(x,y)$$

holds uniformly for any positive integer q which has $\le \log^2 y$ distinct prime factors, all of which are $\le y$, and any $x = y^u,\ 1/2 \le u \le \log y$.

At the end of this section we shall prove

Proposition 2. *The estimate (1.4) holds uniformly for $q \le y^{1/2} \le x \le y^2$.*

Using this it is easy to give the

Proof of Theorems 1 and 4: We shall use Proposition 1 with P as the set of all primes $\le y$, except those dividing q.

For $q > y^{1/2}$ take $x_0 = q,\ x_1 = q \max\{y, \log^2 q\}$ and $\lambda = 0$ in Proposition 1. Using the trivial upper bound $\Psi(x', y; b, q) \ \ll\ x'/q$ we obtain the bound

$$\frac{\Psi(x,y;a,q)}{\Psi_q(x,y)} \ \ll\ \frac{1}{q}\left\{\max_{x_0\le x'\le x_1} \frac{x'}{\Psi_q(x',y)} \ +\ \frac{(u_0+1)yx_0}{\overline{u}_1 \Psi_q(x_1,y)}\right\} \ \ll\ \frac{1}{q} \max_{x_0\le x'\le x_1} \frac{x'}{\Psi_q(x',y)},$$

as x_1 was chosen so that $\frac{(u_0+1)}{\overline{u}_1} y x_0 \ll x_1$, which implies (1.10). Now, if $y^{1/2} \leq q \leq y^N$ then $x_1 = qy$, and so Lemma 3, together with (2.2), gives the upper bound here to be $O(1/\phi(q)) = O(\log q/\phi(q)\log y)$, which implies (1.4) for $q > y^{1/2}$.

For $q \leq y^{1/2}$ (so that $|G| = \phi(q)$), take $x_0 = y^{1/2}$, $x_1 = y^2$ and $\lambda = 1/\phi(q)$. Applying Lemma 1 for $k = 1$ and $u = 0$ and 1, we get $\Psi(x_1, P) \geq |P|^2/8$ as $\Psi(1, P) = 1$. Moreover, by Proposition 2,

$$\max_{b \in G} \Psi(x_0, P; b, q) \ + \ \lambda \Psi(x_0, P) \ \ll \ \frac{\Psi(x_0, P)}{\phi(q)} \ \leq \ \frac{y^{1/2}}{\phi(q)}.$$

Thus, as $|P|\log y \asymp y$ by the Prime Number Theorem, we obtain

$$c \frac{|P|(u_0+1)\log y}{\overline{u}_1 \Psi(x_1, P)} \left\{ \max_{b \in G} \Psi(x_0, P; b, q) + \lambda \Psi(x_0, P) \right\} \ \ll \ \frac{\log^2 y}{\phi(q) y^{1/2}}.$$

Thus, by Proposition 1 (with (2.6) multiplied through by $\phi(q)$), and Proposition 2, we get

$$\left| \frac{\Psi(x, y; a, q)}{\Psi_q(x, y)/\phi(q)} - 1 \right| \ \ll \ \frac{\log q}{\log y} + \frac{\log^2 y}{y^{1/2}} \ \ll \ \frac{\log q}{\log y},$$

for all $x \geq y^{1/2}$ and $(a, q) = 1$.

These estimates combine to establish both Theorems.

Remark 1: If we can obtain a particularly accurate estimate for $\Psi(x, y; a, q)$ in the range $y \leq x \leq y^2$ then we can improve the error term in (1.4), by Proposition 1. For instance, if q is less than some fixed power of $\log y$ then we can apply the Siegel–Walfisz Theorem, and so obtain the estimate (1.2) for all $x \geq y \geq 2$. If $q \leq \exp(\sqrt{\log y})$ then, by expressing the terms $\pi(x/d; q, a/d)$ of (4.4) below as sums over zeros of the Dirichlet L–functions (mod q), we get a large amount of cancellation that enables us to prove a stronger estimate than (1.4) for all $x \geq y \geq 2$. Such results are already obtained in [FT], and so we do not pursue this here.

Remark 2: One reason that we are unable to estimate $\Psi(x, y; a, q)$ when q is much larger than y, is that we have no good method to estimate $|G|$. Trivially, $|G| \geq \Psi_q(q, y)$; if this were equality, then we'd expect that $\Psi(x, y; a, q) \sim \Psi_q(x, y)/\Psi_q(q, y)$ for each $a \in G$, which is $\frac{\Psi_q(x,y)}{\phi(q)} \Big/ \left(\Psi_q(x',y)/\frac{\phi(q)}{q}x'\right)$, evaluated at $x'=q$. This perhaps explains the form of the bound in Theorem 4.

Proof of Theorem 3: Recently Balog and Pomerance [BP] gave the estimate $\Psi(x,y;a,q) = \frac{1}{q}\Psi(x,y)u^{-u\{1+o(1)\}}$, in the intersection of the ranges (1.3) and (1.9) for any $N<4/3$.

We shall take P to be the set of all primes $\leq y$, except those dividing q,

$$x_0 = y^3,\ x_1 = y^5 \quad \text{and} \quad \lambda = \max_{\substack{x_0\leq x'\leq x_1\\ b\in G}} \frac{\Psi(x',P;b,q)}{\Psi(x',P)}$$

in Proposition 1. From Balog and Pomerance's result, we see that λ is bounded above by an absolute constant times $1/\phi(q)$, so that the final term in (2.6) is

$$c\,\frac{|P|(u_0+1)\log y}{\overline{u}_1\Psi(x_1,P)}\left\{\max_{b\in G}\ \Psi(x_0,P;b,q)+\lambda\Psi(x_0,P)\right\} \ll \frac{y\Psi_q(y^3,y)}{\phi(q)\Psi_q(y^5,y)} \ll \frac{1}{y\phi(q)},$$

by Lemma 3. Thus, by Proposition 1 we see that, for any $x\geq x_0$ and $(a,q)=1$, we have

$$\frac{\Psi(x,y;a,q)}{\Psi_q(x,y)} \geq \min_{\substack{x_0\leq x'\leq x_1\\ b\in G}} \frac{\Psi(x',y;b,q)}{\Psi_q(x',y)} - O\left(\frac{1}{y\phi(q)}\right) \gg \frac{1}{\phi(q)},$$

using Balog and Pomerance's result in the last step.

We now proceed to the proof of Proposition 2, but first we need the following easy Lemma:

Lemma 4. *For all positive integers q,*

$$\sum_{d\leq q,\ (d,q)=1} \frac{1}{d} = \frac{\phi(q)}{q}\log q + O\big(\omega(q)\big)\ .$$

Proof: Define $S(x)$ to be the number of integers $\leq x$, that are coprime to q, and let $E(x):= S(x)-\frac{\phi(q)}{q}x$. By partial summation we get, for $x_q = 2^{\omega(q)}$,

$$\begin{aligned}\sum_{d\leq q,\ (d,q)=1} \frac{1}{d} &= \int_{1/2}^{q} \frac{dS(t)}{t} = \Big[\frac{S(t)}{t}\Big]_{1/2}^{q} + \int_{1/2}^{q} \frac{S(t)}{t^2}\,\mathrm{d}t\\ &= \frac{\phi(q)}{q} + \frac{\phi(q)}{q}\int_{1/2}^{q}\frac{\mathrm{d}t}{t} + \int_{1/2}^{x_q}\frac{E(t)}{t^2}\,\mathrm{d}t + \int_{x_q}^{q}\frac{E(t)}{t^2}\,\mathrm{d}t\\ &= \frac{\phi(q)}{q}\,\big(\log q + O(1)\big) + O(\log x_q + 1)\end{aligned}$$

using the trivial facts that $E(t) \le t$ and $E(t) \le x_q$ in the last two integrals. The result follows immediately.

Remark: The Prime Number Theorem implies that $\omega(q) \ll \log q/\log\log q$ and $\phi(q)/q \gg 1/\log\log q$, and so Lemma 4 implies

$$\sum_{d\le q,\ (d,q)=1} \frac{1}{d} \ll \frac{\phi(q)}{q}\log q. \tag{4.1}$$

Proof of Proposition 2: If $y^{1/2} \le x \le y$ then, trivially, $\Psi(x,y;a,q)$ and $\frac{1}{\phi(q)}\Psi_q(x,y)$ can both be estimated by $\frac{x}{q} + O(1)$. Thus (1.4) holds as $1 \ll \frac{x}{q}\frac{\log q}{\log y}$ for all $q \le y^{1/2}$.

Henceforth we will assume that $y \le x \le y^2$. Now, $\Psi(x,x;a,q) - \Psi(x,y;a,q)$ counts the number of integers $n \le x$ of the form $n = pd$ where p is prime, $p > y$ and $p \not| q$, and $d \equiv a/p \pmod{q}$. We count these integers n by first considering those with $p \le x/z$ and then the others (where $z = \min(q, x/y)$), so that

$$\begin{aligned}\Psi(x,y;a,q) &= \Psi(x,x;a,q) - \sum_{\substack{y<p\le x/z\\ p \text{ prime}, p\not| q}} \#\{d \le x/p : d \equiv a/p \pmod{q}\}\\ &\quad - \sum_{\substack{d\le z\\ (d,q)=1}} \{\pi(x/d;q,a/d) - \pi(x/z;q,a/d)\}\end{aligned} \tag{4.2}$$

(note that the first sum is empty if $z = x/y$). Now $\Psi(x,x;a,q) = x/q + O(1)$, and $\#\{d \le x/p : d \equiv a/p \pmod q\} = x/pq + O(1)$ for any prime p that does not divide q. Also

$$\pi(x/d;q,a/d) \ll \frac{x}{\phi(q) d \log(x/qd)}$$

for any $d \le z$ with $(d,q) = 1$, by the Brun-Titchmarsh Theorem; and $\log(x/qd) \asymp \log x$ for any $d \le z$, as $q \le y^{1/2}$. Substituting these estimates into (4.2) gives

$$\begin{aligned}\Psi(x,y;a,q) &= \frac{x}{q}\Big(1 - \sum_{\substack{y<p\le x/z\\ p \text{ prime}, p\not| q}} \frac{1}{p}\Big) + O\Big(\pi\left(\frac{x}{q}\right) + \sum_{\substack{d\le z\\ (d,q)=1}} \frac{x}{\phi(q) d\log x}\Big)\\ &= \frac{x}{q}\Big(1 - \sum_{\substack{y<p\le x/z\\ p \text{ prime}, p\not| q}} \frac{1}{p}\Big) + O\left(\frac{x}{q}\frac{\log q}{\log x}\right) \qquad (4.3)\end{aligned}$$

by the Prime Number Theorem and (4.1). Summing this over all a coprime to q with $1 \le a \le q-1$, and then dividing by $\phi(q)$, we see that (4.3) is also an estimate for $\Psi_q(x,y)/\phi(q)$. Therefore

$$\Psi(x,y;a,q) \;=\; \frac{1}{\phi(q)}\,\Psi_q(x,y) \;+\; O\left(\frac{x}{q}\,\frac{\log q}{\log x}\right),$$

and the result follows as $\Psi_q(x,y) \asymp \frac{\phi(q)}{q}x$ for $y \le x \le y^2$, by Lemma 3.

Remark: Taking $z = x/y$ in (4.2) we see that we need to estimate

$$\sum_{\substack{d \le x/y \\ (d,q)=1}} \{\pi(x/d;q,a/d) - \pi(y;q,a/d)\}.$$

Usually one attacks such estimates by first solving the corresponding problem with "π" replaced by "ψ" (that is the sum of $\log p$ over the prime powers p^m in the arithmetic progression under consideration); and then deducing an answer to the original problem through partial summation. Here, when q is a small fixed power of y, we can obtain an extremely accurate estimate with "π" replaced by "ψ", but are unable to convert this to an accurate estimate for the original problem. Brian Conrey gave the following explanation for this surprising phenomenon:

Take $y = 1$ in the sum above (when "π" is replaced by "ψ") to obtain

$$\begin{aligned}\sum_{\substack{d \le x \\ (d,q)=1}} \psi(x/d;q,a/d) &= \sum_{\substack{d \le x \\ (d,q)=1}} \sum_{\substack{p^m \le x/d \\ p^m \equiv a/d \pmod q}} \log p \\ &= \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \sum_{p^m \mid n} \log p \;=\; \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \log n,\end{aligned}$$

where we take $n = dp^m$; it is easy to estimate this sum with an error term $O(\log x)$. On the other hand, if we do the same series of steps for π then we obtain the identity

$$\sum_{\substack{d \le x \\ (d,q)=1}} \pi(x/d;q,a/d) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \omega(n), \tag{4.4}$$

which is far harder to estimate as accurately.

This leads to the further observation that a "good" uniform estimate for $\Psi(x,y;a,q)$ is (essentially) equivalent to a "good" uniform estimate for $\sum_{\substack{n\le x\\ n\equiv a \pmod q}} \omega(n)$, in the range $qy \le x \le y^2$: To see this, add (4.2) for $z=x/y$ to (4.4), and swap the order of summation for d and p, to obtain,

$$\begin{aligned}\Psi(x,y;a,q)+\sum_{\substack{n\le x\\ n\equiv a \pmod q}}\omega(n) &= \Psi(x,x;a,q)+\sum_{p\le y,\, p\nmid q}\#\{d\le x/p : d\equiv a/p \pmod q\}\\ &= \frac{x}{q}\Big\{1+\sum_{p\le y,\, p\nmid q}\frac{1}{p}\Big\}+O\left(\frac{y}{\log y}\right)\ ,\end{aligned}$$

in the range $y^2\ge x\ge y$ for each $(a,q)=1$. Summing over the arithmetic progressions $a \pmod q$ with $(a,q)=1$, and dividing through by $\phi(q)$, we see that the right side of this equation also serves as an estimate for

$$\frac{1}{\phi(q)}\Big\{\Psi_q(x,y)+\sum_{\substack{n\le x\\ (n,q)=1}}\omega(n)\Big\}\ ,$$

and so

$$\left|\Psi(x,y;a,q)-\frac{1}{\phi(q)}\ \Psi_q(x,y)\right| = \left|\sum_{\substack{n\le x\\ n\equiv a \pmod q}}\omega(n)-\frac{1}{\phi(q)}\sum_{\substack{n\le x\\ (n,q)=1}}\omega(n)\right|+O\left(\frac{y}{\log y}\right)\ .$$

5. Estimates on average — The Proof of Theorem 2.

We now give an upper bound for $\Psi(x,P)$ for all $x\ge y$, using a modification of the method of section 3.

Lemma 5. *Fix $\varepsilon > 0$. The estimate*

$$\Psi(x,P) \ll c_P \Psi(x,y), \qquad \text{where } c_P := \prod_{p\le y,\, p\notin P}\left(1 - \frac{1}{p}\right) \tag{5.1}$$

holds uniformly for all sets of primes P, each $\le y$, and for all $x \ge y^{\varepsilon}$.

Proof: Let $x_0 = y^{\varepsilon}$ and $x_1 = 2yx_0$, so that $\Psi(x_1,y) \gg \Psi(x_0,y)|P|\log y$, by (2.2). An easy consequence of the Fundamental Lemma of Sieve Theory ([JR]), is that

$$\Psi(x,P) \le \sum_{\substack{n\le x\\ (n,q)=1}} 1 \ll \frac{\phi(q)}{q}x \ll c_P x,$$

holds uniformly for all $x \ge x_0$, where q is the product of the primes $\le x$ that are not in the set P. This implies (5.1) for $x_0 \le x \le x_1$, by (2.2).

For each $x \ge x_0$ define $r(x) = \Psi(x,P)/c_P\Psi(x,y)$ and $r^*(x) = \max_{x_0\le x'\le x} r(x)$, so that $r^*(x_1)$ is bounded. We will proceed in a similar manner to the proof of Proposition 1:

When $x \ge 2x_0$, we can give an upper bound on $\Psi(x,P)$ by bounding each term of the form $\Psi(t,P)$ on the right side of (3.2) in the following, trivial, way:

$$\Psi(t,P) \le \begin{cases} \Psi(x_0,P) = r(x_0)c_P\Psi(x_0,y), & \text{for } 1 \le t \le x_0;\\ r^*(x/2)c_P\Psi(t,y), & \text{for } x_0 \le t \le x/2;\\ r^*(x)c_P\Psi(t,y), & \text{for } x/2 \le t \le x. \end{cases}$$

This leads to the formula

$$r(x) \le \frac{1}{2}r^*(x/2) + \frac{1}{2}r^*(x) + r(x_0)(|P|+1)\frac{\Psi(x_0,y)\log(x_0y)}{\Psi(x,y)\log x},$$

which may be compared to (3.4); and then to

$$r^*(x) \le r^*(x/2) + r(x_0)(|P|+1)\frac{\Psi(x_0,y)\log(x_0y)}{\Psi(x/2,y)\log(x/2)}.$$

From this we deduce that

$$r^*(x) \le r^*(x_1) + O\left(r(x_0)|P|\frac{\Psi(x_0,y)\log y}{\Psi(x_1,y)}\right),$$

which we already know to be bounded.

In order to prove Theorem 2, we shall need the following lemma which may be of independent interest (this is based on Lemma 1 of [HB]):

Lemma 6. *For any given sequence of integers, and positive integer q, define $D_q(x)$ to be the maximum, over those integers a that are coprime to q, of the absolute value of the difference between the number of elements of the sequence $\leq x$ that belong to the congruence class $a \pmod q$ and $\frac{1}{\phi(q)}$ times the number of elements of the sequence $\leq x$ that are coprime to q. Suppose that for all fixed $A > 0$ there exists a constant $B_1 = B_1(A) > 0$ such that*

$$\sum_{q \leq x^{1/2}/L^{B_1}} D_q(x) \ll_A \frac{x}{L^A},$$

where, for this lemma, $L := \log x$. Then, for each fixed $A > 0$, there exists a constant $B_2 = B_2(A) > 0$ such that

$$\sum_{q \leq x^{1/2}/L^{B_2}} \max_{x' \leq x} D_q(x') \ll_A \frac{x}{L^A}.$$

Proof: Let $E = A+1$, $C = B_1(3E)$ and $B = B_2(A) := C + E/2 + 1$. Choose any $Q \leq x^{1/2}/2L^B$ and let $M = [L^E]$. Choose integer $x_q \leq x$ so that $D_q(x_q)$ is maximized and then take $j = [x_q M/x]$. As there are $\leq y/q + 1$ integers in any arithmetic progression $\pmod q$ in any interval of length y, we have $D_q(x_q) \leq D_q(jx/M) + x/Mq + 1$. Therefore, as $D_q(jx/M) \ll x/Q$ for any q in the range $Q < q \leq 2Q$, we obtain

$$\begin{aligned}
\sum_{Q<q\leq 2Q} D_q(x_q)^2 &\ll \sum_{Q<q\leq 2Q} \left(\frac{x}{Mq}\right)^2 + \sum_{j=1}^{M} \sum_{Q<q\leq 2Q} D_q(jx/M)^2 \\
&\ll \frac{x^2}{L^{2E}Q} + \frac{x}{Q} \sum_{j=1}^{M} \sum_{Q<q\leq 2Q} D_q(jx/M) \\
&\ll \frac{x^2}{L^{2E}Q} + \frac{x}{Q}\, M\, \frac{x}{L^{3E}} \ll \frac{x^2}{L^{2E}Q}
\end{aligned}$$

by the hypothesis, as $2Q \leq (x/M)^{1/2}/L^C$. Then, by the Cauchy–Schwarz inequality,

$$\sum_{Q<q\leq 2Q} D_q(x_q) \ll \left(Q \sum_{Q<q\leq 2Q} D_q(x_q)^2\right)^{1/2} \ll \frac{x}{L^E},$$

and the result follows from summing over the relevant such diadic intervals up to $2Q = x^{1/2}/L^{B_2}$.

Using this, we can now give the

Proof of Theorem 2: In [FT], Fouvry and Tenenbaum proved that for any given $E > 0$ there exists a constant $B_1 = B_1(E) > 0$ such that the estimate

$$\sum_{q \leq x^{1/2}/\log^{B_1} x} \max_{(a,q)=1} \left| \Psi(x, y; a, q) - \frac{\Psi_q(x, y)}{\phi(q)} \right| \ll_E \frac{x}{\log^E x}$$

holds uniformly for $x \geq y \geq 2$. Then, by Lemma 6, for any given $A' > 0$ there exists a constant $B_2 = B_2(A') > 0$ such that the estimate

$$\sum_{q \leq x^{1/2}/\log^{B_2} x} \max_{x' \leq x} \max_{(a,q)=1} \left| \Psi(x', y; a, q) - \frac{\Psi_q(x', y)}{\phi(q)} \right| \ll_{A'} \frac{x}{\log^{A'} x} \tag{5.2}$$

holds uniformly for $x \geq y \geq 2$.

Let $A' = A + 3C + 3$ and $B = 2B_2(A') + 2$. Let $P = P_q$ be the set of all primes $\leq y$ except those dividing q, and define Γ_q and Γ_q^* to be as in the proof of Propsition 1, with $\lambda = 1/\phi(q)$. For any y and Q in the range (1.7) let $u_1 = 3C\log_2 y/\log_3 y$ and $x_1 = y^{u_1}$, and $x_0 = y^{u_0} = \max\{Q^2\log^B Q, y^{2u_1/3}\}$.

It is well known (see [dB] for instance) that if $u > 1$ then $\log \rho(u) \sim -u\log u$. Thus, if $x \leq x_1$ then, by (2.2),

$$\Psi(x, y) \gg x/\log^{3C+o(1)} y. \tag{5.3}$$

Therefore (5.2) and (5.3) imply (1.6) for $Q^2\log^B Q \leq x \leq x_1$.

Let $N = [\log y]$ and $z_i = x_0(x_1/x_0)^{i/N}$ for $i = 0, 1, \ldots N$. By Lemma 3 and (5.2), we get

$$\frac{\Psi(z_i, y)}{Q} \sum_{Q < q \leq 2Q} \max_{z_i < z \leq z_{i+1}} \phi(q)\Gamma_q(z) \ll \sum_{Q < q \leq 2Q} \max_{z_i < z \leq z_{i+1}} \Gamma_q(z)\Psi_q(z, y) \ll_A \frac{z_{i+1}}{\log^{A'} y},$$

and so, as $z_{i+1}/\Psi(z_i,y) \ll \log^{3C+o(1)} y$ by (5.3), thus

(5.4)
$$\frac{1}{Q}\sum_{Q<q\le 2Q} \phi(q)\Gamma_q^*(x_1) \le \sum_{i=0}^{N-1}\frac{1}{Q}\sum_{Q<q\le 2Q}\max_{z_i\le z\le z_{i+1}} \phi(q)\Gamma_q(z) \ll \frac{1}{\log^{A'-3C-1+o(1)} y}.$$

Now, using Lemma 3 and (5.3) to estimate each $\Psi(x_1,P)$, and also using the trivial bounds $|G|\le\phi(q)$, $|P|\log y \ll y$ and $\Psi(x_0,P;b,q)\ll x_0/q$ we obtain, for the final term in (2.6),

$$c\,\frac{|P|(u_0+1)\log y}{\overline{u}_1\Psi(x_1,P)}\left\{\max_{b\in G}\ \Psi(x_0,P;b,q)+\frac{\Psi(x_0,P)}{\phi(q)}\right\} \ll \frac{yx_0}{\phi(q)\Psi(x_1,y)} \ll \frac{1}{\phi(q)y}.$$

Therefore, by Lemma 5, we get that if $x_0\le x'\le x$ then

(5.5)
$$\Psi_q(x',y)\Gamma_q(x') \le \Psi_q(x,y)\Gamma_q^*(x) \ll \frac{\Psi(x,y)}{Q}\phi(q)\Gamma_q^*(x) \ll \frac{\Psi(x,y)}{Q}\left(\phi(q)\Gamma_q^*(x_1)+\frac{1}{y}\right),$$

using Proposition 1.

Now, by summing (5.5) over all q in the range $Q<q\le 2Q$, and by using (5.4) to estimate the resulting sum, we obtain

$$\sum_{Q<q\le 2Q}\max_{x_0\le x'\le x}\max_{(a,q)=1}\left|\Psi(x',y;a,q)-\frac{\Psi_q(x',y)}{\phi(q)}\right| \ll \frac{\Psi(x,y)}{\log^{A+2+o(1)} y}.$$

Finally, if we sum this over suitable diadic intervals and add the result to (5.2) with $x=x_0$ (in case the maximum occurs for some $x\le x_0$ for some q in the sum), we obtain the Theorem.

6. Some remarks and extensions.

For any set of primes P and modulus q we are able to prove that (2.5) holds once x is sufficiently large; specifically, if z is the smallest positive integer for which $\Psi(z,P;a,q)\ge 1$

for all $a \in G$, then we can show that (2.5) holds for $|P|\log z + |P|^2\log y = o(\log x)$. We conjecture that $\log z \ll |G|\log y/|P|$ (which leads to the range $|G| + |P|^2 = o(u)$ in (2.5)); currently, we can only prove that $\log z \ll |G|\log y$ (which leads to the range $|G||P| = o(u)$ for $q \geq y$ in (2.5)). In general one might expect that x can be taken to be substantially smaller than this; but, one can construct examples where (2.5) fails to hold with $|P|\log z = \log x$. However, in such examples, P tends to be a fairly 'thin' set of primes, and so one might conjecture that if P contains a 'reasonable' proportion of the primes $\leq y$ then (2.5) holds uniformly when $\log(x/z)/\log y \to \infty$ as $x \to \infty$.

An important shortcoming of the method presented here is that the error term doesn't shrink as x gets larger (though it doesn't get much bigger, either). However, given that Selberg examined similar functional equations in his elementary proof of the Prime Number Theorem for arithmetic progressions [Se], we might hope that a suitable modification of the proof of Proposition 1 will allow us to improve the error term in (1.4) as u gets larger; this viewpoint will be explored further in the sequel.

It is straightforward to prove that the estimate

$$\Psi(x, y; a, q) = \frac{\Psi_q(x, y)}{\phi(q)} \left\{ 1 + O\left(\frac{q\log^2 y}{y^{1/2}} \right) \right\} \tag{6.1}$$

holds uniformly for all $x \geq y \geq 2$ with $q \leq y^{1/2}/\log^2 y$ under the assumption of the Generalized Riemann Hypothesis: A well known consequence of the Generalized Riemann Hypothesis is the estimate

$$\pi(x; q, a) = \frac{\pi(x)}{\phi(q)} + O(x^{1/2}\log^2 qx).$$

We substitute this into (4.2) with $z = x/y$ for $y^{1/2} \leq x \leq y^{3/2}/\log^3 y$ and $z = x^{1/3}/\log^2 x$ for $y^{3/2}/\log^3 y \leq x \leq y^2$, to obtain (6.1) for $y^{1/2} \leq x \leq y^2$. The result then follows by Proposition 1 with $\lambda = 1/\phi(q)$. A similar result can be obtained by the method of [FT] though with the restriction that $y \geq \log^{2+\varepsilon} x$. By more careful considerations, along the lines of [BP], we might hope to further extend the range for q in which (1.1) is valid, under the assumption of the Generalized Riemann Hypothesis as well as certain conjectures about Kloosterman sums.

7. Related problems.

Our iteration method can be used to attack many related questions. For instance, we can show that

$$\Psi(x,y;\chi) := \sum_{\substack{n\le x \\ p|n\Rightarrow p\le y}} \chi(n) \ll \Psi_q(x,y)\,\frac{\log q}{\log y} \tag{7.1}$$

holds uniformly for any non–principal character $\chi(\bmod\ q)$ in the range (1.5). This is proved by first showing (7.1) in the range $y^{1/2}\le x\le y^2$, and then by establishing a result similar to Proposition 1, which comes from comparing the identity

$$\Psi(x,y;\chi)\log x = \int_1^x \frac{\Psi(t,y;\chi)}{t}\mathrm{d}t + \sum_{\substack{p\le y,\, p^m\le x \\ p\nmid q}} \Psi\left(\frac{x}{p^m},y;\chi\right)\chi(p^m)\log p$$

with (3.2). We thus extend the estimate $\Psi(x,y;\chi)=o(\Psi_q(x,y))$ to a much wider range than previously known, although Theorem 4 of [FT] provides a much sharper estimate in a limited range.

Another interesting function to examine is $\Psi(x,y,\mu) := \sum_{n\le x, p|n\Rightarrow p\le y}\mu(n)$, where $\mu(n)$ is the Möbius function, which plays a prominent rôle in Daboussi's recent proof of the Prime Number Theorem [Db]. Alladi [Al] showed that

$$\Psi(x,y,\mu) \ll \Psi(x,y)/\log x \tag{7.2}$$

for all $x\ge y\ge \exp((\log x)^{5/8+\varepsilon})$, and later extended this to the range (2.4). In [H3], Hildebrand proved a somewhat stronger estimate for $y\le \exp((\log x)^{1/21})$, which allowed him to extend the range of (7.2) to all $x\ge y\ge 2$; and Tenenbaum [T2] has recently given a superior estimate for all of the range $x \geq y^{1+\varepsilon}$, using the saddle point method. We can use our methods to show that $\Psi(x,y,\mu) \ll \Psi(x,y)/\log^2 y$ for all $x \geq y^2$ with $y \geq 2$, by first showing this in the range $y^2 \leq x \leq y^4$ (see [Al], Theorem 2), and then from a result similar to Proposition 1, which is obtained from comparing the identity

$$\Psi(x,y,\mu)\log x \;=\; \int_1^x \frac{\Psi(t,y,\mu)}{t}\mathrm{d}t \;-\; \sum_{\substack{p^m \leq x \\ p \leq y}} \Psi\Big(\frac{x}{p^m}, y, \mu\Big)\log p$$

with (2.3).

A similar method works for the corresponding sum of Liouville's function, $\lambda(n)$, and indeed of many multiplicative functions, $f(n)$, with values inside or on the unit circle. Recently Hildebrand [H4] and Elliott [El] have given strong results of this type, using the large sieve.

Our method may also be applied to Selberg's formula [Se], to show that if an estimate of the form

$$\left|\pi(x;q,a) - \frac{\pi(x)}{\phi(q)}\right| \;\leq\; \Delta\frac{\pi(x)}{\phi(q)}$$

holds whenever $(a,q)=1$ for, say, $x_0 \leq x \leq x_1$, then a slightly weaker bound holds for all $x \geq x_0$.

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