RESEARCH ANNOUNCEMENT: FASTER FACTORIZATION INTO COPRIMES

DANIEL J. BERNSTEIN

Abstract. This paper presents an algorithm that, given positive integers a, b, computes the natural coprime base for $\{a, b\}$ in time $n(\lg n)^{2+o(1)}$, where n is the number of input bits. This paper also presents an algorithm that, given a set S of positive integers, computes the natural coprime base for S in time $n(\lg n)^{4+o(1)}$.

1. Introduction

My previous paper [1] introduced an algorithm that, given a set S of positive integers, computes the natural coprime base $\operatorname{cb} S$ in time $n(\lg n)^{O(1)}$, where n is the number of input bits. I made no attempt in [1] to optimize the exponent of $\lg n$.

Section 2 of this paper presents an algorithm that computes $\operatorname{cb}\{a, b\}$ in time $n(\lg n)^{2+o(1)}$. It is reasonable to conjecture that the limiting exponent 2 is optimal (for, e.g., a multitape Turing machine): one has $\operatorname{cb}\{a, b\} = \{a, b\} - \{1\}$ if and only if a, b are coprime; the well-known problem of checking coprimality has been stuck at $n(\lg n)^{2+o(1)}$ for thirty years.

Section 4 of this paper presents an algorithm that computes $\operatorname{cb}(\{a\} \cup Q)$, where Q is any coprime set, in time $n(\lg n)^{2+o(1)}$. Substitute this algorithm into Algorithms 17.3 and 18.1 of [1] to obtain $\operatorname{cb}(P \cup Q)$ in time $n(\lg n)^{3+o(1)}$ and $\operatorname{cb} S$ in time $n(\lg n)^{4+o(1)}$. I'm not willing to conjecture that the 3 and 4 are optimal.

This is a very early draft. I'm confident in the basic structure of the algorithms, but there could be some silly omissions, and of course the proofs need vastly more detail.

2. Computing a coprime base for two positive integers

The following algorithm computes $\operatorname{cb}\{a, b\}$, given positive integers a and b, in time $n(\lg n)^{2+o(1)}$.

Step 1. Swap a, b if necessary so that $a \geq b$. The algorithm will later reduce the input length by at least one third of the length of a. If $a = 1$, stop.

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Step 2. Compute $a_0 = a$, $g_0 = \gcd\{a_0, b\}$, $a_1 = a_0/g_0$, $g_1 = \gcd\{a_1, g_0^2\}$, $a_2 = a_1/g_1$, $g_2 = \gcd\{a_2, g_1^2\}$, and so on, until $g_k = 1$.

For example, if $a = 2^{100}3^{100}$ and $b = 2^{137}3^{13}$, compute $a_0 = 2^{100}3^{100}$, $g_0 = 2^{100}3^{13}$, $a_1 = 3^{87}$, $g_1 = 3^{26}$, $a_2 = 3^{61}$, $g_2 = 3^{52}$, $a_3 = 3^9$, $g_3 = 3^9$, $a_4 = 1$, $g_4 = 1$.

Lower level: The gcd inputs a_i, g_{i-1}^2 are often highly unbalanced. To compute $\gcd\{a_i, g_{i-1}^2\}$, first divide a_i by g_{i-1}^2, and then use any standard fast gcd algorithm to compute $\gcd\{g_{i-1}^2, a_i \bmod g_{i-1}^2\}$. The division takes time $n(\lg n)^{1+o(1)}$; the gcd takes time $m(\lg m)^{2+o(1)}$ where m is the length of g_{i-1}^2.

All the g's together have length $O(n)$, and k is at most about $\lg n$, so the total time here is $n(\lg n)^{2+o(1)}$.

Step 3. Compute $x_0 = g_0/\gcd\{g_0, g_1^\infty\}$, $x_1 = g_1/\gcd\{g_1, g_2^\infty\}$, and so on.

For example, if $a = 2^{100}3^{100}$ and $b = 2^{137}3^{13}$, compute $x_0 = 2^{100}$, $x_1 = 1$, $x_2 = 1$, $x_3 = 3^9$.

Lower level: Compute each $\gcd\{g_{i-1}, g_i^\infty\}$ as $\gcd\{g_{i-1}, g_i^{2^{e_i}} \bmod g_{i-1}\}$ where e_i is the smallest nonnegative integer satisfying $2^{2^{e_i}} \ge g_{i-1}$. The repeated squarings and gcd take time $m(\lg m)^{2+o(1)}$ where m is the total length of g_{i-1}, g_i. The total time here is $n(\lg n)^{2+o(1)}$.

Step 4. Compute $y_0 = \gcd\{b, x_0^\infty\}$, $y_1 = \gcd\{g_0, x_1^\infty\}$, $y_2 = \gcd\{b, g_1, x_2^\infty\}$, $y_3 = \gcd\{b, g_2, x_3^\infty\}$, and so on.

For example, if $a = 2^{100}3^{100}$ and $b = 2^{137}3^{13}$, the algorithm computes $y_0 = 2^{137}$, $y_1 = 1$, $y_2 = 1$, $y_3 = 3^{13}$.

Lower level: Use a scaled remainder tree to compute $b \bmod g_1, b \bmod g_2, \ldots$; this takes time $n(\lg n)^{2+o(1)}$ since $b, g_1, g_2, \ldots$ together have length $O(n)$. Then compute $\gcd\{b, g_1\}$ as $\gcd\{b \bmod g_1, g_1\}$; compute $\gcd\{b, g_2\}$ as $\gcd\{b \bmod g_2, g_2\}$; and so on.

Step 5. Recursively print $\mathrm{cb}\{x_0, y_0/x_0\}$; $\mathrm{cb}\{x_1, y_1\}$; $\mathrm{cb}\{x_2, y_2\}$; and so on. Also print $\mathrm{cb}\{a'\} = \{a'\} - \{1\}$ and $\mathrm{cb}\{b'\} = \{b'\} - \{1\}$ where $a' = a/\gcd\{a, b^\infty\}$ and $b' = b/\gcd\{b, a^\infty\}$. Note that a' has already been computed; it equals a_k.

For example, if $a = 2^{100}3^{100}$ and $b = 2^{137}3^{13}$, recursively print $\mathrm{cb}\{2^{100}, 2^{37}\} = \{2\}$ and $\mathrm{cb}\{3^9, 3^{13}\} = \{3\}$. Also print $\mathrm{cb}\{1\} = \{\}$ and $\mathrm{cb}\{1\} = \{\}$. The complete output is $\{2, 3\}$.

I claim that $x_0y_0, x_1y_1, \ldots, a', b'$ are coprime; that $a = a'x_0x_1y_1x_2y_2^3x_3y_3^7\cdots$; that $b = b'y_0y_1y_2y_3\cdots$; and that $y_0x_1y_1x_2y_2\cdots$, the product of inputs to the recursive calls, is at most $ab/a^{1/3} \le (ab)^{5/6}$. Each of these facts can be checked from the following table of ord_p values, expressed in terms of $e = \mathrm{ord}_p\, a$ and $f = \mathrm{ord}_p\, b$:

g_0	g_1	g_2	g_3	$\ldots$	x_0	y_0	x_1	y_1	x_2	y_2	$\ldots$	a'	b'	
0	0	0	0	$\ldots$	0	0	0	0	0	0	$\ldots$	0	f	if $e = 0$
e	0	0	0	$\ldots$	e	f	0	0	0	0	$\ldots$	0	0	if $0 < e \le f$
f	$e-f$	0	0	$\ldots$	0	0	$e-f$	f	0	0	$\ldots$	0	0	if $f < e \le 3f$
f	$2f$	$e-3f$	0	$\ldots$	0	0	0	0	$e-3f$	f	$\ldots$	0	0	if $3f < e \le 7f$
														$\vdots$
0	0	0	0	$\ldots$	0	0	0	0	0	0	$\ldots$	e	0	if $f = 0 < e$

Consequently the outputs of the algorithm are coprime; a and b are products of powers of the outputs; and the recursion multiplies the total time by a bounded factor.

Note that one can easily factor a, b over $\mathrm{cb}\{a, b\}$ by tracing the factorizations $a = a' x_0 x_1 y_1 x_2 y_2^3 x_3 y_3^7 \cdots$ and $b = b' y_0 y_1 y_2 y_3 \cdots$ through the recursion.

3. An algorithm without a catchy name

The following algorithm computes $\gcd\{s, p^\infty\}$ for each s in a multiset S and for each p in a nonempty coprime set P. It takes time $(k+1)n(\lg n)^{2+o(1)}$ if $\#P \le 2^k$.

See [2] and [3] for similar algorithms.

Step 1. If $\#P = 1$: Find $p \in P$. Use a scaled remainder tree to compute $p \bmod s$ for each $s \in S$. Compute $\gcd\{s, p^\infty\}$ as $\gcd\{s, (p \bmod s)^\infty\}$. This takes time $n(\lg n)^{2+o(1)}$.

Assume from now on that $\#P \ge 2$.

Step 2. Select $Q \subseteq P$ with $\#Q = \lfloor \#P/2 \rfloor$. Use a product tree to compute $y = \prod_{p \in Q} p$. This takes time $n(\lg n)^{2+o(1)}$.

Step 3. Use a scaled remainder tree to compute $y \bmod s$ for each $s \in S$. This takes time $n(\lg n)^{2+o(1)}$.

Step 4. Compute $\gcd\{s, (y \bmod s)^\infty\} = \gcd\{s, y^\infty\}$ for each $s \in S$. This takes time $n(\lg n)^{2+o(1)}$.

Step 5. Apply the algorithm recursively to $\{\gcd\{s, y^\infty\} : s \in S\}$ and Q; separately handle each s for which $\gcd\{s, y^\infty\} = 1$. This produces $\gcd\{\gcd\{s, y^\infty\}, p^\infty\} = \gcd\{s, p^\infty\}$ for each $p \in Q$.

Apply the algorithm recursively to $\{s/\gcd\{s, y^\infty\} : s \in S\}$ and $P - Q$; separately handle each s for which $s/\gcd\{s, y^\infty\} = 1$. This produces $\gcd\{s/\gcd\{s, y^\infty\}, p^\infty\} = \gcd\{s, p^\infty\}$ for each $p \in P - Q$.

The product of inputs at each level of recursion is exactly the original product of inputs, so the total input size at each level of recursion is $O(n)$.

4. Extending a coprime base

The following algorithm computes $\mathrm{cb}(\{a\} \cup Q)$, where Q is coprime, in time $n(\lg n)^{2+o(1)}$.

Step 1. Use a product tree to compute $b = \prod_{q \in Q} q$. This takes time $n(\lg n)^{2+o(1)}$.

Step 2. Define, and compute, $a_0, g_0, a_1, g_1, \ldots, x_0, x_1, \ldots, y_0, y_1, \ldots, a'$ exactly as in Section 2. This takes time $n(\lg n)^{2+o(1)}$.

Step 3. Write $y_i(q)$ for $\gcd\{q, y_i^\infty\}$. Compute $y_0(q), y_1(q), y_2(q), \ldots$ for each $q \in Q$ as explained in Section 3. This takes time $n(\lg n)^{2+o(1)} \lg\lg n = n(\lg n)^{2+o(1)}$. Check for and discard 1's so that they do not slow down subsequent computations.

Step 4. Use product trees to compute $z_0 = \prod_q y_0(q)$, $z_1 = \prod_q y_1(q)$, etc. This takes time $n(\lg n)^{2+o(1)}$.

Notice that $z_1 z_2^3 z_3^7 \cdots$ divides a. Indeed, take any prime p dividing $z_1 z_2^3 z_3^7 \cdots$. Recall that $y_1, y_2, \ldots$ are coprime, so p divides z_i for a unique i. Write $e = \mathrm{ord}_p\, a$ and $f = \mathrm{ord}_p\, b$; then $(2^i - 1)f < e \le (2^{i+1} - 1)f$. Furthermore, p divides a unique $q \in Q$, and $f = \mathrm{ord}_p\, q = \mathrm{ord}_p\, z_i$ by definition of b and z_i, so $(2^i - 1)\,\mathrm{ord}_p\, z_i < \mathrm{ord}_p\, a$.

Step 5. Use a scaled remainder tree to compute $a \bmod z_0, a \bmod z_1^3, a \bmod z_2^7, \ldots$. This takes time $n(\lg n)^{2+o(1)}$, since the product $z_1^3 z_2^7 \cdots$ divides a^3.

Step 6. Use scaled remainder trees to compute $(a \bmod z_0) \bmod y_0(q) = a \bmod y_0(q)$ for each q; $(a \bmod z_1^3) \bmod y_1(q)^3 = a \bmod y_1(q)^3$ for each q; $(a \bmod z_2^7) \bmod y_2(q)^7 = a \bmod y_2(q)^7$ for each q; etc. This takes time $n(\lg n)^{2+o(1)}$.

Step 7. Compute $\gcd\{a, y_0(q)\}$, $\gcd\{a, y_1(q)^3\}$, $\gcd\{a, y_2(q)^7\}$, etc. This takes time $n(\lg n)^{2+o(1)}$.

Observe that $\gcd\{a, y_0(q)\}$, $\gcd\{a, y_1(q)^3\}$, $\gcd\{a, y_2(q)^7\}$, etc. are the same as $\gcd\{a, y_0(q)^\infty\}$, $\gcd\{a, y_1(q)^\infty\}$, $\gcd\{a, y_2(q)^\infty\}$, etc. Consider, for example, a prime p dividing $\gcd\{a, y_2(q)^\infty\}$. Recall that $3f < e \le 7f$ where $e = \operatorname{ord}_p a$ and $f = \operatorname{ord}_p b = \operatorname{ord}_p y_2(q)$; thus $\operatorname{ord}_p \gcd\{a, y_2(q)^\infty\} = e = \operatorname{ord}_p \gcd\{a, y_2(q)^7\}$.

Step 8. Print $\operatorname{cb}\{y_i(q), \gcd\{a, y_i(q)^\infty\}\}$ for each i and q. Also print $q/y_0(q)y_1(q)\cdots$ for each q, and print a'. This takes time $n(\lg n)^{2+o(1)}$.

References

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[2] Daniel J. Bernstein, *How to find small factors of integers*, accepted to Mathematics of Computation; now being revamped. URL: `http://cr.yp.to/papers.html`.

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Department of Mathematics, Statistics, and Computer Science (M/C 249), The University of Illinois at Chicago, Chicago, IL 60607–7045

E-mail address: `djb@cr.yp.to`