DETECTING PERFECT POWERS BY FACTORING INTO COPRIMES

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Abstract. This paper presents an algorithm that, given an integer $n > 1$, finds the largest integer k such that n is a kth power. A previous algorithm by the first author took time $b^{1+o(1)}$ where $b = \lg n$; more precisely, time $b \exp(O(\sqrt{\lg b \lg \lg b}))$; conjecturally, time $b(\lg b)^{O(1)}$. The new algorithm takes time $b(\lg b)^{O(1)}$. It relies on relatively complicated subroutines—specifically, on the first author's fast algorithm to factor integers into coprimes—but it allows a proof of the $b(\lg b)^{O(1)}$ bound without much background; the previous proof of $b^{1+o(1)}$ relied on transcendental number theory.

Here is an algorithm that, given an integer $n > 1$, finds the largest integer k such that n is a kth power:

1. For each prime power q such that $2^q \le n$, write down a positive integer r_q such that if n is a qth power then $n = r_q^q$.
2. Find a finite coprime set P of integers larger than 1 such that each of $n, r_2, r_3, r_4, r_5, r_7, \dots$ is a product of powers of elements of P. (In this paper, "coprime" means "pairwise coprime.")
3. Factor n as $\prod_{p \in P} p^{n_p}$, and compute $k = \gcd\{n_p : p \in P\}$.

It is easy to see that the algorithm is correct. Say n is an ℓth power. Take any prime power q dividing ℓ. Then n is a qth power, so $n = r_q^q$; but r_q is a product $\prod_{p \in P} p^{a_p}$ for some exponents a_p, so n is a product $\prod_{p \in P} p^{q a_p}$. Factorizations over P are unique, so $n_p = q a_p$ for each p. Thus q divides $\gcd\{n_p : p \in P\} = k$. This is true for all q, so ℓ divides k. Conversely, n is certainly a kth power.

Take, for example, $n = 49787136 < 2^{26}$. Compute approximations

$$
\begin{array}{lll}
r_2 = 7056 \approx n^{1/2} & r_8 = 9 \approx n^{1/8} & r_{17} = 3 \approx n^{1/17} \\
r_3 = 368 \approx n^{1/3} & r_9 = 7 \approx n^{1/9} & r_{19} = 3 \approx n^{1/19} \\
r_4 = 84 \approx n^{1/4} & r_{11} = 5 \approx n^{1/11} & r_{23} = 2 \approx n^{1/23} \\
r_5 = 35 \approx n^{1/5} & r_{13} = 4 \approx n^{1/13} & r_{25} = 2 \approx n^{1/25} \\
r_7 = 13 \approx n^{1/7} & r_{16} = 3 \approx n^{1/16} &
\end{array}
$$

where $\approx$ means "within 0.6." Factor $\{49787136, 7056, 368, 84, 35, 13, 9, 7, 5, 4, 3, 2\}$ into coprimes: each of these numbers is a product of powers of elements of $P = \{2, 3, 5, 7, 13, 23\}$. In particular, $n = 2^8 3^4 5^0 7^4 13^0 23^0$, so $k = \gcd\{8, 4, 0, 4, 0, 0\} = 4$. In other words, n is a 4th power, and is not an ℓth power for $\ell > 4$.

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As discussed below, the literature already shows how to perform each step of this algorithm in time $b(\lg b)^{O(1)}$, where $b = \lg n$. Computing $n^{1/k}$, which is used by some applications, also takes time $b(\lg b)^{O(1)}$.

Details of Step 1. Given n and q, one can use binary search and Newton's method to compute a floating-point number guaranteed to be within (e.g.) 2^{-32} of $n^{1/q}$, as explained in [4, Sections 8 and 10]. The algorithms of [4] rely on FFT-based integer multiplication; see [6, Sections 2–4].

Define r_q as an integer within 2^{-32} of this floating-point number. If no such integer exists, define $r_q = 1$.

Each r_q has $O(b/q)$ bits. Together the r_q's have $\sum_{q \le \lg n} O(b/q) = O(b \lg\lg b)$ bits by Mertens's theorem. The algorithms of [4] take time $(\lg b)^{O(1)}$ per bit.

An alternative approach to Step 1 is to define r_q as an integer 2-adically close to $n^{1/q}$, as explained in [4, Section 21].

Details of Step 2. Given a finite set of positive integers, the algorithm of [5, Section 18] computes the "natural coprime base" for that set. The algorithm takes time $s(\lg s)^{O(1)}$ where s is the number of input bits. The algorithm relies on FFT-based multiplication, division, and gcd; see [6, Sections 17 and 22].

Use this algorithm to compute the "natural coprime base" P for $\{n, r_2, \dots\}$. Together $n, r_2, \dots$ have $O(b \lg\lg b)$ bits, so this takes time $b(\lg b)^{O(1)}$.

Details of Step 3. Given a finite coprime set P of integers larger than 1, and given a positive integer that has a factorization over P, the algorithm of [5, Section 20] finds that factorization. The algorithm takes time $s(\lg s)^{O(1)}$ where s is the number of input bits. The algorithm relies on FFT-based arithmetic.

Use this algorithm to factor n over P. Together n and P have $O(b \lg\lg b)$ bits, so this takes time $b(\lg b)^{O(1)}$.

Competition. Previous work by the first author in [4] had already shown that k could be computed in time $b^{1+o(1)}$. The algorithm of [4] computes r_q for prime numbers q, and then computes several increasingly precise approximations to r_q^q, stopping when an approximation demonstrates that $r_q^q \ne n$.

The run-time bound for the algorithm in this paper has two advantages over the run-time bound for the algorithm in [4]:

- The new bound is smaller. The old bound was $b \exp(O(\sqrt{\lg b \lg\lg\lg b}))$; the new bound is $b(\lg b)^{O(1)}$.
- The new proof requires considerably less background. The new proof relies on the first author's results in [5] on factoring into coprimes, but the old proof relied on deep results in transcendental number theory.

The old algorithm is conjectured to take time $b(\lg b)^{O(1)}$, as discussed in [4, Section 15], but this conjecture seems very difficult to prove.

For typical n's, both algorithms see from the initial approximation to $n^{1/q}$ that n is not a qth power; the bottleneck is the computation of that approximation. The algorithms behave differently only in the atypical case that n is very close to a power. Optimization of the worst case in more detail than $b(\lg b)^{O(1)}$ is beyond the scope of this paper.

History. Bach, Driscoll, and Shallit in [2] introduced a quadratic-time algorithm to factor integers into coprimes. The obvious algorithm takes cubic time.

Bach and Sorenson in [3] published various algorithms to detect perfect powers, i.e., to check whether $k > 1$. One algorithm takes time $O(b^3)$. Another algorithm is conjectured to take time $O(b^2/(\lg b)^2)$ for most, but not all, n's.

The second and third authors of this paper observed in early 1994 that they could compute k in time $O(b^2(\lg\lg b)^2)$ by factoring $n, r_2, \ldots$ into coprimes with the Bach-Driscoll-Shallit algorithm; recall that $n, r_2, \ldots$ together have $O(b \lg\lg b)$ bits. This line of work was abandoned several months later when the first author announced that k could be computed in time $b^{1+o(1)}$ by the increasingly-precise-approximations-to-r_q^q method.

The first author later pointed out that this line of work deserved to be revived, since he had found an essentially-linear-time algorithm—see [5]—to factor integers into coprimes.

References

[1] Eric Bach, James Driscoll, Jeffrey Shallit, *Factor refinement*, in [7] (1990), 201–211; see also newer version [2]. URL: `http://cr.yp.to/bib/entries.html#1990/bach-cba`.

[2] Eric Bach, James Driscoll, Jeffrey Shallit, *Factor refinement*, Journal of Algorithms **15** (1993), 199–222; see also older version [1]. ISSN 0196–6774. MR 94m:11148. URL: `http://cr.yp.to/bib/entries.html#1993/bach-cba`.

[3] Eric Bach, Jonathan Sorenson, *Sieve algorithms for perfect power testing*, Algorithmica **9** (1993), 313–328. ISSN 0178–4617. MR 94d:11103.

[4] Daniel J. Bernstein, *Detecting perfect powers in essentially linear time*, Mathematics of Computation **67** (1998), 1253–1283. ISSN 0025–5718. MR 98j:11121. URL: `http://cr.yp.to/papers.html`.

[5] Daniel J. Bernstein, *Factoring into coprimes in essentially linear time*, to appear, Journal of Algorithms. ISSN 0196–6774. URL: `http://cr.yp.to/papers.html`. ID `f32943f0bb67a9317d4021513f9eee5a`.

[6] Daniel J. Bernstein, *Fast multiplication and its applications*, to appear in Buhler-Stevenhagen *Algorithmic number theory* book. URL: `http://cr.yp.to/papers.html#multapps`.

[7] David S. Johnson (editor), *Proceedings of the first annual ACM-SIAM symposium on discrete algorithms, January 22–24, 1990, San Francisco, California*, Society for Industrial and Applied Mathematics, Philadelphia, 1990. ISBN 0–89871–251–3.

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