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SHARPER ABC-BASED BOUNDS FOR CONGRUENT POLYNOMIALS

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ABSTRACT. Agrawal, Kayal, and Saxena recently introduced a new method of proving that an integer is prime. The speed of the Agrawal-Kayal-Saxena method depends on proven lower bounds for the size of the multiplicative semigroup generated by several polynomials modulo another polynomial h. Voloch pointed out an application of the Stothers-Mason ABC theorem in this context: under mild assumptions, distinct polynomials A, B, C of degree at most $1.2 \deg h - 0.2 \deg \operatorname{rad} ABC$ cannot all be congruent modulo h. This paper presents two improvements in the combinatorial part of Voloch's argument. The first improvement moves the degree bound up to $2 \deg h - \deg \operatorname{rad} ABC$. The second improvement generalizes to $m \geq 3$ polynomials $A_1, \dots, A_m$ of degree at most $((3m-5)/(3m-7)) \deg h - (6/(3m-7)m) \deg \operatorname{rad} A_1 \cdots A_m$.

1. Introduction

Fix a nonconstant univariate polynomial h over a field k. Assume that the characteristic of k is at least $3 \deg h - 1$. The main theorem of this paper, Theorem 2.3, states that if $m \geq 3$ distinct polynomials $A_1, \dots, A_m$ are all congruent modulo h and coprime to h then

$$\max\{\deg A_1, \dots, \deg A_m\} > \frac{3m-5}{3m-7} \deg h - \frac{6}{(3m-7)m} \deg \operatorname{rad} A_1 \cdots A_m.$$

As usual, $\operatorname{rad} X$ means the largest monic squarefree divisor of X, i.e., the product of the monic irreducibles dividing X. If $\deg \operatorname{rad} A_1 \cdots A_m < (m/3) \deg h$ then this bound is better than the obvious bound $\max\{\deg A_1, \dots, \deg A_m\} > \deg h - 1$.

For example, if distinct polynomials A, B, C are congruent modulo h and coprime to h then $\max\{\deg A, \deg B, \deg C\} > 2 \deg h - \deg \operatorname{rad} ABC$. No better bound is possible in this level of generality: if $h = x^{10} - 1$, $A = x^{20}$, $B = x^{10}$, and $C = 1$ then $\operatorname{rad} ABC = \operatorname{rad} x^{30} = x$ so $2 \deg h - \deg \operatorname{rad} ABC = 19$.

The proof relies on the Stothers-Mason ABC theorem. Analogous bounds in the number-field case follow from the ABC conjecture.

Previous work. Voloch in [3] proved that $\max\{\deg A, \deg B, \deg C\} > 1.2 \deg h - 0.2 \deg \operatorname{rad} ABC$. This paper improves Voloch's result in two ways: first, it is quantitatively stronger, in the interesting case that $\deg \operatorname{rad} ABC < \deg h$; second, it applies to larger values of m.

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Application. Consider the subgroup G of $(k[x]/h)^*$ generated by $\{x - s : s \in S\}$, where $S \subseteq k$ and $0 \notin h(S)$. The Agrawal-Kayal-Saxena primality-proving method requires a lower bound on $\#G$ for groups G of this type, typically with $\#S = \deg h$. The primality-proving method becomes faster as the lower bound on $\#G$ increases, as discussed in [1, Section 7].

This paper shows that $\#G \geq \frac{1}{m-1}\binom{\lfloor((3m-5)/(3m-7))\deg h-(6/(3m-7)m)\#S\rfloor+\#S}{\#S}$ for any $m \geq 3$. Indeed, the binomial coefficient is the number of products of powers of $\{x - s\}$ in $k[x]$ of degree at most $\lfloor((3m-5)/(3m-7))\deg h - (6/(3m-7)m)\#S\rfloor$; m distinct such products cannot all have the same image modulo h.

In particular, if $\#S = \deg h$, then $\#G \geq \frac{1}{3}\binom{\lfloor 2.1\deg h\rfloor}{\deg h} \approx 4.27689^{\deg h}$. Compare this to the bound $\#G \geq \binom{2\deg h-1}{\deg h} \approx 4^{\deg h}$ obtained from a degree bound of $\deg h - 1$. Note that the improvement requires $m > 3$.

Different methods from [3] produce a lower bound around $5.828^{\deg h}$, so the ABC-based techniques in [3] and in this paper have not yet had an impact on the speed of primality proving. However, I suspect that these techniques have not yet reached their limits.

2. Proofs

Theorem 2.1. *Let k be a field. Let h be a positive-degree element of the polynomial ring $k[x]$. Assume that $1, 2, 3, \ldots, 3\deg h - 2$ are invertible in k. Let A, B, C be distinct nonzero elements of $k[x]$. If $\gcd\{A, B, C\} = 1$ and $A \equiv B \equiv C \pmod h$ then $\max\{\deg A, \deg B, \deg C\} > 2\deg h - \deg\operatorname{rad} ABC$.*

Proof. Assume without loss of generality that $\deg A = \max\{\deg A, \deg B, \deg C\}$. The nonzero polynomial $A-B$ is a multiple of h, so $\deg A \geq \deg(A-B) \geq \deg h > 0$; thus $\deg\operatorname{rad} ABC > 0$.

If $\deg A \geq 2\deg h$ then $\deg A > 2\deg h - \deg\operatorname{rad} ABC$; done.

Define $U = (B-C)/h$, $V = (C-A)/h$, and $W = (A-B)/h$. Then $U \neq 0$; $V \neq 0$; $W \neq 0$; U, V, W each have degree at most $\deg A - \deg h$; and $UA + VB + WC = 0$. Define $D = \gcd\{UA, VB, WC\}$.

If $\deg D = \deg UA$ then UA divides VB, WC; so A divides VWA, VWB, VWC; so A divides $\gcd\{VWA, VWB, VWC\} = VW$; but $VW \neq 0$, so $\deg A \leq \deg VW \leq 2(\deg A - \deg h)$; so $\deg A \geq 2\deg h$; done.

Assume from now on that $\deg D < \deg UA$ and that $\deg A \leq 2\deg h - 1$. Then $\deg(UA/D)$ is between 1 and $2\deg A - \deg h \leq 3\deg h - 2$; so the derivative of UA/D is nonzero. Also $UA/D + VB/D + WC/D = 0$, and $\gcd\{UA/D, VB/D, WC/D\} = 1$. By Theorem 3.1 below, $\deg(UA/D) < \deg\operatorname{rad}((UA/D)(VB/D)(WC/D))$.

The proof follows Voloch up to this point. Voloch next observes that D divides $\gcd\{UVWA, UVWB, UVWC\} = UVW\gcd\{A, B, C\} = UVW$. I claim that more is true: $D\operatorname{rad}((UA/D)(VB/D)(WC/D))$ divides $UVW\operatorname{rad} ABC$.

(In other words: If $d = \min\{u + a, v + b, w + c\}$ and $\min\{a, b, c\} = 0$ then $d + [u + v + w + a + b + c > 3d] \leq u + v + w + [a + b + c > 0]$. Proof: Without loss of generality assume $a = 0$. Then $d \leq u \leq u + v + w$. If $d < u + v + w$ then $d + [\cdots] \leq d + 1 \leq u + v + w \leq u + v + w + [\cdots]$ as claimed. If $a + b + c > 0$ then $d + [\cdots] \leq u + v + w + 1 = u + v + w + [\cdots]$ as claimed. Otherwise $u + v + w + a + b + c = d \leq 3d$ so $d + [u + v + w + a + b + c > 3d] = d \leq u + v + w \leq u + v + w + [\cdots]$ as claimed.)

Thus $\deg UA < \deg(D \operatorname{rad}((UA/D)(VB/D)(WC/D))) \le \deg(UVW \operatorname{rad} ABC)$. Hence $\deg A < \deg(VW \operatorname{rad} ABC) \le 2(\deg A - \deg h) + \deg \operatorname{rad} ABC$; i.e., $\deg A > 2\deg h - \deg \operatorname{rad} ABC$ as claimed. $\square$

Theorem 2.2. *Let k be a field. Let h be a positive-degree element of the polynomial ring $k[x]$. Assume that $1, 2, 3, \ldots, 3\deg h - 2$ are invertible in k. Let A, B, C be distinct nonzero elements of $k[x]$. If $\gcd\{A, B, C\}$ is coprime to h and $A \equiv B \equiv C \pmod h$ then*

$$\begin{aligned}\max\{\deg A, \deg B, \deg C\}& \\ &> 2\deg h - \deg \operatorname{rad} A - \deg \operatorname{rad} B - \deg \operatorname{rad} C \\ &\quad + \deg \operatorname{rad} \gcd\{A, B\} + \deg \operatorname{rad} \gcd\{A, C\} + \deg \operatorname{rad} \gcd\{B, C\}.\end{aligned}$$

Proof. Write $G = \gcd\{A, B, C\}$. Then G is coprime to h, so $A/G \equiv B/G \equiv C/G \pmod h$. By Theorem 2.1,

$$\max\left\{\deg \frac{A}{G}, \deg \frac{B}{G}, \deg \frac{C}{G}\right\} > 2\deg h - \deg \operatorname{rad} \frac{ABC}{GGG} \ge 2\deg h - \deg \operatorname{rad} ABC,$$

so $\max\{\deg A, \deg B, \deg C\} \ge 2\deg h + \deg G - \deg \operatorname{rad} ABC$. But $\deg G \ge \deg \operatorname{rad} G = \deg \operatorname{rad} ABC - \deg \operatorname{rad} A - \deg \operatorname{rad} B - \deg \operatorname{rad} C + \deg \operatorname{rad} \gcd\{A, B\} + \deg \operatorname{rad} \gcd\{A, C\} + \deg \operatorname{rad} \gcd\{B, C\}$ by inclusion-exclusion. $\square$

Theorem 2.3. *Let k be a field. Let h be a positive-degree element of the polynomial ring $k[x]$. Assume that $1, 2, 3, \ldots, 3\deg h - 2$ are invertible in k. Let S be a finite subset of $k[x] - \{0\}$, with $\#S \ge 3$. If each element of S is coprime to h, and all the elements of S are congruent modulo h, then*

$$\max\{\deg A : A \in S\} > \frac{3\#S - 5}{3\#S - 7} \deg h - \frac{6}{(3\#S - 7)\#S} \deg \operatorname{rad} \prod_{A \in S} A.$$

For example, $\max\{\deg A : A \in S\} > 1.4 \deg h - 0.3 \deg \operatorname{rad} \prod_{A \in S} A$ if $\#S = 4$, and $\max\{\deg A : A \in S\} > 1.25 \deg h - 0.15 \deg \operatorname{rad} \prod_{A \in S} A$ if $\#S = 5$.

Proof. Define $d = \max\{\deg A : A \in S\}$ and $e = \deg \operatorname{rad} \prod_{A \in S} A$. By Theorem 2.2,

$$\begin{aligned}d &> 2\deg h - \deg \operatorname{rad} A - \deg \operatorname{rad} B - \deg \operatorname{rad} C \\ &\quad + \deg \operatorname{rad} \gcd\{A, B\} + \deg \operatorname{rad} \gcd\{A, C\} + \deg \operatorname{rad} \gcd\{B, C\}\end{aligned}$$

for any distinct $A, B, C \in S$. Average this inequality over all choices of A, B, C to see that $d > 2\deg h - 3 \operatorname{avg}_A \deg \operatorname{rad} A + 3 \operatorname{avg}_{A \ne B} \deg \operatorname{rad} \gcd\{A, B\}$. On the other hand, $e \ge \#S \operatorname{avg}_A \deg \operatorname{rad} A - \binom{\#S}{2} \operatorname{avg}_{A \ne B} \deg \operatorname{rad} \gcd\{A, B\}$ by inclusion-exclusion, so

$$d + \frac{3}{\#S} e > 2\deg h - \frac{3\#S - 9}{2} \operatorname{avg}_{A \ne B} \deg \operatorname{rad} \gcd\{A, B\}.$$

Note that $3\#S - 9 \ge 0$ since $\#S \ge 3$.

One can bound each term $\deg \operatorname{rad} \gcd\{A, B\}$ by the simple observation that $A/\gcd\{A, B\}$ and $B/\gcd\{A, B\}$ are distinct congruent polynomials of degree at most $d - \deg \gcd\{A, B\}$; thus $d - \deg \gcd\{A, B\} \ge \deg h$, so $\deg \operatorname{rad} \gcd\{A, B\} \le d - \deg h$. Hence

$$d + \frac{3}{\#S} e > 2\deg h - \frac{3\#S - 9}{2}(d - \deg h);$$

i.e., $d > ((3\#S - 5)/(3\#S - 7)) \deg h - (6/(3\#S - 7)\#S) e$. $\square$

3. Appendix: the ABC theorem

Theorem 3.1 is a typical statement of the Stothers-Mason ABC theorem, included in this paper for completeness. The proof given here is due to Noah Snyder; see [2].

Theorem 3.1. *Let k be a field. Let A, B, C be nonzero elements of the polynomial ring $k[x]$ with $A+B+C=0$ and $\gcd\{A,B,C\}=1$. If $\deg A \geq \deg \operatorname{rad} ABC$ then $A'=0$.*

In fact, $A'=B'=C'=0$. As usual, X' means the derivative of X; the relevance of derivatives is that $X/\operatorname{rad} X$ divides X'.

Proof. Note that $\gcd\{A,B\} = \gcd\{A,B,-(A+B)\} = \gcd\{A,B,C\} = 1$. By the same argument, $\gcd\{A,C\}=1$ and $\gcd\{B,C\}=1$.

$C/\operatorname{rad} C$ divides both C and C', so it divides $C'B - CB'$. Similarly, $B/\operatorname{rad} B$ divides $C'B-CB'$. Furthermore, $C' = -(A'+B')$, so $C'B-CB' = -(A'+B')B + (A+B)B' = AB' - A'B$; thus $A/\operatorname{rad} A$ divides $C'B - CB'$.

The ratios $A/\operatorname{rad} A, B/\operatorname{rad} B, C/\operatorname{rad} C$ are pairwise coprime, so their product $ABC/\operatorname{rad} ABC$ divides $C'B - CB'$. But by hypothesis $\deg(ABC/\operatorname{rad} ABC) = \deg ABC - \deg\operatorname{rad} ABC \geq \deg BC > \deg(C'B - CB')$; so $C'B - CB' = 0$; so $AB' - A'B = 0$; so A divides $A'B$; but A and B are coprime, so A divides A'; but $\deg A > \deg A'$, so $A' = 0$. $\square$

References

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