Proceedings of the American Mathematical Society **121** (1994), 405–408.

# A NON-ITERATIVE 2-ADIC STATEMENT OF THE $3N+1$ CONJECTURE

Daniel J. Bernstein

9/21/92

Abstract. Associated with the $3N+1$ problem is a permutation $\Phi$ of the 2-adic integers. The $3N+1$ conjecture is equivalent to the conjecture that $3Q$ is an integer if $\Phi(Q)$ is a positive integer. We state a new definition of $\Phi$. To wit: $Q$ and $N=\Phi(Q)$ are linked by the equations $Q = 2^{d_0} + 2^{d_1} + \cdots$ and $N = (-1/3)2^{d_0} + (-1/9)2^{d_1} + (-1/27)2^{d_2} + \cdots$ with $0 \le d_0 < d_1 < \cdots$. We list four applications of this definition.

## Definition and conjecture

We recall that the 2-adic integers $\mathbf{Z}_2$ may be defined as binary expansions which are allowed to extend infinitely far to the left [3, Ex. 4.1–31]: for instance, $1/3 = (\ldots 0101011)_2 \in \mathbf{Z}_2$. Fix odd 2-adic integers $u_0, u_1, \ldots \in 1 + 2\mathbf{Z}_2$. For any increasing, finite or infinite, sequence $0 \le d_0 < d_1 < \cdots$ of nonnegative integers, the sum $\sum u_i 2^{d_i}$ converges to some 2-adic integer $S$. In fact this map from increasing sequences to sums is one-to-one and onto all of $\mathbf{Z}_2$. We construct its inverse. Given $S \in \mathbf{Z}_2$ we set $d_0$ to the first bit position in $S$, $d_1$ to the first bit position in $S - u_0 2^{d_0}$, and so on. If at any point $S - u_0 2^{d_0} - \cdots - u_n 2^{d_n}$ is zero, we stop, and the sequence is finite.

As an example, the expansion

$$Q = 2^{d_0} + 2^{d_1} + \cdots \tag{1}$$

is a bijection between $Q \in \mathbf{Z}_2$ and increasing sequences $d = \langle d_0, d_1, \ldots \rangle$. Finite sequences $d$ correspond to nonnegative integers $Q$. (In particular the empty sequence corresponds to $Q = 0$.) Similarly, the expansion

$$N = \frac{-1}{3}2^{d_0} + \frac{-1}{9}2^{d_1} + \frac{-1}{27}2^{d_2} + \cdots \tag{2}$$

is another bijection. Together (1) and (2) determine a bijection between all $N \in \mathbf{Z}_2$ and all $Q \in \mathbf{Z}_2$. We write $N = \Phi(Q)$. For instance,

$$\Phi(-\frac{1}{3}) = \Phi(2^0 + 2^2 + 2^4 + \cdots) = \frac{-1}{3}2^0 + \frac{-1}{9}2^2 + \frac{-1}{27}2^4 + \cdots = 1 \tag{3}$$

by elementary geometric series manipulations. So $1 \in \Phi((1/3)\mathbf{Z})$.

**Conjecture.** *The set $\mathbf{Z}^+$ of positive integers is contained in $\Phi((1/3)\mathbf{Z})$.*

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1991 *Mathematics Subject Classification.* Primary 11D72.

The author was supported in part by a National Science Foundation Graduate Fellowship. He would also like to thank Bellcore for the use of their facilities in preparing this document.

## Connection with the $3N+1$ problem

We define two functions $H(Q)$ and $C(N)$. If $Q$ is even we set $H(Q) = Q/2$; otherwise we set $H(Q) = Q - 1$. If $N$ is even we set $C(N) = N/2$; otherwise we set $C(N) = 3N + 1$.

**Theorem 1.** *$C(\Phi(Q)) = \Phi(H(Q))$.*

*Proof.* We define $d$ as in (1). If $Q$ is even then $d_0 > 0$ (or $d$ has length 0) and

$$C(\Phi(Q)) = \frac{-1}{3}2^{d_0-1} + \frac{-1}{9}2^{d_1-1} + \cdots = \Phi(2^{d_0-1} + 2^{d_1-1} + \cdots) = \Phi(Q/2).$$

If $Q$ is odd then $d_0 = 0$ and

$$\begin{aligned} C(\Phi(Q)) &= 1 + 3(\frac{-1}{3} + \frac{-1}{9}2^{d_1} + \frac{-1}{27}2^{d_2}\cdots) = \frac{-1}{3}2^{d_1} + \frac{-1}{9}2^{d_2} + \cdots \\ &= \Phi(2^{d_1} + 2^{d_2} + \cdots) = \Phi(Q - 2^{d_0}) = \Phi(Q-1) \end{aligned}$$

as desired. □

The $3N+1$ conjecture [4] states that, for any positive integer $N \in \mathbf{Z}^+$, some iterate $C^k(N)$ equals 1. This implies our conjecture:

**Theorem 2.** *If $C^k(N) = 1$ then $N \in \Phi((1/3)\mathbf{Z})$.*

*Proof.* Set $Q = \Phi^{-1}(N)$. Now $C^k(\Phi(Q)) = 1$ so $\Phi(H^k(Q)) = 1$ so $H^k(Q) = \Phi^{-1}(1) = -1/3 \in (1/3)\mathbf{Z}$. If $H(x) \in (1/3)\mathbf{Z}$ then $x \in (1/3)\mathbf{Z}$, so by induction $Q \in (1/3)\mathbf{Z}$ as desired. □

The converse is also true: our conjecture implies the $3N+1$ conjecture.

**Theorem 3.** *If $N \in \mathbf{Z}^+$ and $N \in \Phi((1/3)\mathbf{Z})$ then $C^k(N) = 1$ for some $k$.*

*Proof.* Again set $Q = \Phi^{-1}(N)$ and define $d$ by (1) and (2). We have $3Q \in \mathbf{Z}$. Notice first that $Q$ cannot be an integer. For if $Q \in \mathbf{Z}$ then either $Q = 0$, in which case $N = 0$; or $Q$ is positive, in which case $d$ is finite and $N$ is a negative rational number by (2); or $Q$ is negative, in which case $d_{i+1} = d_i + 1$ for all large $i$ and so (2) again converges to a negative rational number.

So $Q$ is $1/3$ away from an integer. Thus the $d_i$'s eventually fall into the pattern $d_{i+1} = d_i + 2$, say for $i \geq m$. Define a map $c$ on $d$ corresponding to the action of $C$ on $N$. Notice that $c$ acts on $d$ by subtracting 1 from each element, if $d_0 > 0$; or by shifting $d$ to the left, if $d_0 = 0$. So

$$c^{d_m+m}(\langle d_0, \ldots, d_{m-1}, d_m, d_m+2, d_m+4, \ldots\rangle) = \langle 0, 2, 4, \ldots\rangle,$$

and $C^{d_m+m}(N) = 1$. □

Hence our conjecture about the 2-adic expansions (1) and (2) is equivalent to the $3N+1$ conjecture. Does this throw any light on the latter? Our map $\Phi$ is exactly the inverse of $Q_\infty$ in [4]. (This can alternatively be derived from Lemma 4 in [5].) Our Theorem 1, that $\Phi$ conjugates $H$ to $C$, is equivalent to Theorem 1 of [1]. What is new here is the expansion (2). It gives an explicit formula for $\Phi = Q_\infty^{-1}$. We have therefore answered affirmatively the final question in [4].

## Applications

Our result has several immediate applications. First, say $Q \in \mathbf{Q} \cap \mathbf{Z}_2$ is rational. Then either $d$ is finite (say of length $\mu$) or $d_{m+\lambda} = d_m + X$ for all sufficiently large $m$ (say $m \geq \mu$) and some fixed $\lambda$ and $X$. In the first case $3^\mu N$ is an integer. In the second case

$$-3^\mu(3^\lambda - 2^X)N = (3^\lambda - 2^X)(3^{\mu-1}2^{d_0} + \cdots + 3^0 2^{d_{\mu-1}}) + (3^{\lambda-1}2^{d_\mu} + \cdots + 3^0 2^{d_{\mu+\lambda-1}})$$

by (2). So in either case $N$ is rational.

**Corollary 1.** $\Phi(\mathbf{Q} \cap \mathbf{Z}_2) \subseteq \mathbf{Q} \cap \mathbf{Z}_2$.

The "Periodicity Conjecture" from [4] states that $\Phi(\mathbf{Q} \cap \mathbf{Z}_2) = \mathbf{Q} \cap \mathbf{Z}_2$. We have shown half of this.

Second, in our development of $\Phi$ we noted that $\Phi$ is a bijection from $\mathbf{Z}_2$ onto itself, i.e., a permutation of $\mathbf{Z}_2$. In fact we see from (1) and (2) that $\Phi$ is a homeomorphism under the topology induced by the usual 2-adic metric (see [4, section 2.8]). So Theorem L of [4] follows immediately.

Third, from (1) and (2) we see that the function $Q_k(N) = \Phi^{-1}(N) \bmod 2^k$ depends only on the equivalence class $N \bmod 2^k$. This is the first half of Theorem B from [4]. The induced function $\overline{Q}_k$ on equivalence classes is a permutation because $\Phi$ is. By induction the cycles of $\overline{Q}_k$ are of length dividing $2^k$. Indeed, any cycle of $\overline{Q}_k$ of length $r$ gives rise to either one cycle of $\overline{Q}_{k+1}$ of length $2r$ or two cycles of $\overline{Q}_{k+1}$ of length $r$. This is the second half of Theorem B from [4].

Finally, we give a short proof of the following theorem of Müller [5]:

**Corollary 2.** *$\Phi$ is nowhere differentiable.*

*Proof.* If $d$ is infinite then for any $k \geq 0$

$$\frac{\Phi(Q) - \Phi(Q - 2^{d_k})}{2^{d_k}} \equiv (-1)^k \pmod 4$$

by routine computation from (2). So as $k \to \infty$ the difference ratio does not converge. If $d$ is finite, say of length $m$, then for $e + f > e > d_{m-1}$

$$\frac{\Phi(Q) - \Phi(Q + 2^e + 2^{e+f})}{2^e + 2^{e+f}} = \frac{1}{3^{m+2}} \frac{3 + 2^f}{1 + 2^f}$$

again by routine computation; and the latter quantity is different mod 8 for $f = 1, 2$. So as $e \to \infty$ the difference ratio does not converge. Hence both $\Phi$ and its inverse are nowhere differentiable. $\square$

Note that our approach generalizes to the "$AN + B$ problem" [2] [6]. In this generalization $A$ and $B$ are odd, and $N = \Phi_{A,B}(Q) = \sum(-B/A^{i+1})2^{d_i}$. This homeomorphism $\Phi_{A,B}$ conjugates $H$ to $C_{A,B}$, where $C_{A,B}(N)$ equals $N/2$ for $N$ even and $AN + B$ for $N$ odd.

## Acknowledgments

The author would like to thank Jeffrey C. Lagarias and an anonymous referee for their helpful suggestions.

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5 Brewster Lane, Bellport, NY 11713