1999 Putnam problems and unofficial solutions

As usual, first come the problems, then the problems with solutions. Comments and criticism at the end.

Send any followup remarks to the `USENET` newsgroup `sci.math`.

Problems

Problem A1

Find polynomials $f(x)$, $g(x)$ and $h(x)$, if they exist, such that, for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if} \quad x < -1 \\ 3x+2 & \text{if} \quad -1 \le x \le 0 \\ -2x+2 & \text{if} \quad x > 0. \end{cases}$$

Problem A2

Let $p(x)$ be a polynomial that is non-negative for all x. Prove that, for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

Problem A3

Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \ge 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

Problem A4

Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n 3^m + m 3^n)}.$$

Problem A5

Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^{1} |p(x)|\, dx.$$

Problem A6

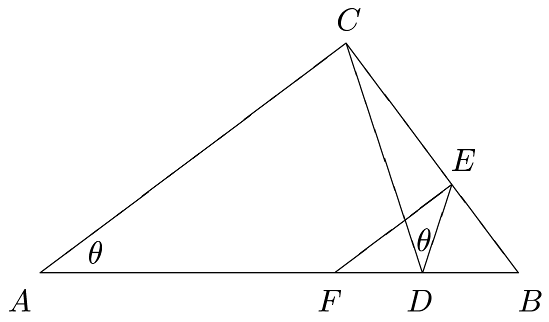
The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n.

Problem B1

Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0} |EF|$. [Here, $|PQ|$ denotes the length of the line segment PQ.]



Problem B2

Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots. [The roots may be either real or complex.]

Problem B3

Let $A = \{(x, y) : 0 \leq x, y < 1\}$. For $(x, y) \in A$, let

$$S(x, y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{\substack{(x,y)\to(1,1)\\(x,y)\in A}} (1-xy^2)(1-x^2y)S(x,y).$$

Problem B4

Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Problem B5

For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.

Problem B6

Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime. [Here, $\gcd(a, b)$ denotes the greatest common divisor of a and b.]

Unofficial solutions

Problem A1

Find polynomials $f(x)$, $g(x)$ and $h(x)$, if they exist, such that, for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if} \quad x < -1 \\ 3x + 2 & \text{if} \quad -1 \leq x \leq 0 \\ -2x + 2 & \text{if} \quad x > 0. \end{cases}$$

Solution: Take $f(x) = (3/2)(x + 1)$, $g(x) = (5/2)x$, and $h(x) = 1/2 - x$.

If $x < -1$ then $|f(x)| = -(3/2)(x + 1)$ and $|g(x)| = -(5/2)x$ so $|f(x)| - |g(x)| + h(x) = -(3/2)x - 3/2 + (5/2)x + 1/2 - x = -1$.

If $-1 \leq x \leq 0$ then $|f(x)| = (3/2)(x+1)$ and $|g(x)| = -(5/2)x$ so $|f(x)| - |g(x)| + h(x) = (3/2)x + 3/2 + (5/2)x + 1/2 - x = 3x + 2$.

If $x > 0$ then $|f(x)| = (3/2)(x + 1)$ and $|g(x)| = (5/2)x$ so $|f(x)| - |g(x)| + h(x) = (3/2)x + 3/2 - (5/2)x + 1/2 - x = -2x + 2$.

Problem A2

Let $p(x)$ be a polynomial that is non-negative for all x. Prove that, for some k, there are polynomials $f_1(x), \dots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

Solution: If p is constant then $p(x) = (p(0)^{1/2})^2$. Otherwise find a root $r + si$ of p.

If $s \neq 0$ then $r - si$ is another root of p, so $p(x)$ is divisible by $(x - r - si)(x - r + si) = (x - r)^2 + s^2$.

If $s = 0$ then $p(x)$ must be divisible by $(x - r)^2$; otherwise $p(r + \epsilon)p(r - \epsilon)$ would be negative for small ϵ.

Either way $p(x) = ((x - r)^2 + s^2)q(x)$ for some polynomial q. Now $(x - r)^2 + s^2$ is positive for all $x \neq r$, so $q(x)$ is nonnegative for all $x \neq r$; by continuity, $q(x)$ is nonnegative for all x. By induction on degree, $q(x) = \sum_j f_j(x)^2$ for some $f_1, f_2, \dots$. Thus $p(x) = \sum_j ((x - r)f_j(x))^2 + \sum_j (sf_j(x))^2$ as desired.

Problem A3

Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

Solution: Write $r = 1 + \sqrt{2}$ and $s = 1 - \sqrt{2}$. Then $rs = -1$ and $r + s = 2$, so

$$\begin{aligned}
\sum_n a_n x^n &= \frac{1}{1 - 2x - x^2} = \frac{1}{(1 - rx)(1 - sx)} = \frac{r/(r - s)}{1 - rx} - \frac{s/(r - s)}{1 - sx} \\
&= \sum_n \frac{r^{n+1} - s^{n+1}}{r - s} x^n.
\end{aligned}$$

Thus $a_n = (r^{n+1} - s^{n+1})/(r - s)$, and

$$\begin{aligned}
a_n^2 + a_{n+1}^2 &= \frac{r^{2n+2} + s^{2n+2} - 2(rs)^{n+1} + r^{2n+4} + s^{2n+4} - 2(rs)^{n+2}}{(r - s)^2} \\
&= \frac{r^{2n+3}(r + 1/r) + s^{2n+3}(s + 1/s) - 2(rs)^{n+1}(1 + rs)}{(r - s)^2} \\
&= \frac{r^{2n+3}(r - s) + s^{2n+3}(s - r) - 2(rs)^{n+1}(1 - 1)}{(r - s)^2} = \frac{r^{2n+3} - s^{2n+3}}{r - s} = a_{2n+2}.
\end{aligned}$$

Another approach: Define $b_n = a_n^2 + a_{n+1}^2$; then $b_0 = a_2$ and $b_1 = a_4$. Starting from the identity $a_{n+2} = 2a_{n+1} + a_n$ one can check that $b_{n+2} = 6b_{n+1} - b_n$ and that $a_{2n+6} = 6a_{2n+4} - a_{2n+2}$, so $b_n = a_{2n+2}$ for all n by induction.

Yet another approach: Convert the identity $a_n = 2a_{n-1} + a_{n-2}$ into the matrix equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} a_n & a_{n-1} \\ a_{n-1} & a_{n-2} \end{pmatrix}$$

and square both sides to obtain

$$\begin{pmatrix} a_{2n} & a_{2n-1} \\ a_{2n-1} & a_{2n-2} \end{pmatrix} = \begin{pmatrix} a_n & a_{n-1} \\ a_{n-1} & a_{n-2} \end{pmatrix}^2 = \begin{pmatrix} a_n^2 + a_{n-1}^2 & a_{n-1}(a_n + a_{n-2}) \\ a_{n-1}(a_n + a_{n-2}) & a_{n-1}^2 + a_{n-2}^2 \end{pmatrix}.$$

Problem A4

Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n3^m + m3^n)}.$$

Solution: All sums here are absolutely convergent.

Write S for the desired sum. Then

$$\begin{aligned} 2S &= \sum_{m,n} \frac{m^2 n}{3^m (n3^m + m3^n)} + \sum_{m,n} \frac{n^2 m}{3^n (m3^n + n3^m)} \\ &= \sum_{m,n} \frac{m^2 n 3^n + n^2 m 3^m}{3^m 3^n (m3^n + n3^m)} = \sum_{m,n} \frac{mn}{3^m 3^n} = T^2 \end{aligned}$$

where $T = \sum_m (m/3^m)$.

If $0 < x < 1$ then $x/(1-x) = \sum_m x^m$. Differentiate: $1/(1-x)^2 = \sum_m m x^{m-1}$. Substitute $x = 1/3$: $9/4 = \sum_m m/3^{m-1} = 3T$. Thus $T = 3/4$ and $S = T^2/2 = 9/32$.

Problem A5

Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^{1} |p(x)| \, dx.$$

Solution: Factor $p(x)$ as $\ell \prod_{1 \leq j \leq 1999} (x - r_j)$.

Define $S_k = \{x : 0.999 - 0.001k < x < 1 - 0.001k\}$. The 2000 sets $S_0, S_1, \ldots, S_{1999}$ are disjoint, so there is some k such that S_k contains none of the real parts of $r_1, \ldots, r_{1999}$. Write $u = 0.9995 - 0.001k$ and $v = 0.9996 - 0.001k$.

Fix $x \in [u, v]$. If $|r_j| \leq 2$ then $|x - r_j| \geq 0.0002 \geq |r_j| / 10000$ by choice of k. If $|r_j| > 2$ then $|x - r_j| \geq |r_j| - 1 \geq |r_j| / 10000$ since $[u, v] \subseteq [-1, 1]$. Consequently $|p(x)| = |\ell| \prod_j |x - r_j| \geq |\ell| \prod_j (|r_j| / 10000) = |p(0)| / 10000^{1999}$.

Thus $\int_{-1}^{1} |p(x)|\, dx \geq \int_u^v |p(x)|\, dx \geq (v - u) |p(0)| / 10000^{1999} = |p(0)| / 10000^{2000}$.

Problem A6

The sequence $(a_n)_{n \geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n.

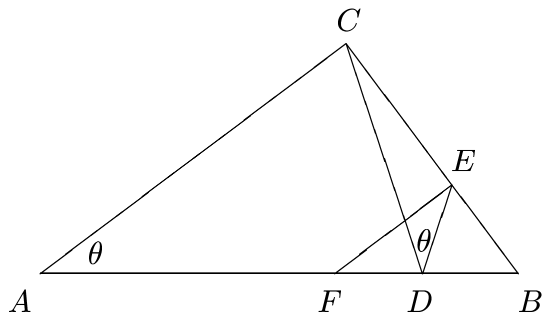
Solution: Define $b_n = a_{n+1}/a_n$, so that $a_n = b_{n-1}b_{n-2} \cdots b_2 b_1$. Then $b_1 = 2$, $b_2 = 12$, and $b_{k+2} = 6b_{k+1} - 8b_k$; by induction $b_k = 4^k - 2^k = (2^k - 1)2^k$.

It suffices to show that $\operatorname{ord}_p a_n \geq \operatorname{ord}_p n$ for all primes p. If p is odd then p divides $2^k - 1$, and hence b_k, whenever $p - 1$ divides k; if $p = 2$ then p divides 2^k, and hence b_k, for all k. In either case

$$\operatorname{ord}_p a_n \geq \left\lfloor \frac{n-1}{p-1} \right\rfloor \geq \left\lfloor \frac{p^{\operatorname{ord}_p n} - 1}{p-1} \right\rfloor \geq \left\lfloor \frac{(p-1)\operatorname{ord}_p n}{p-1} \right\rfloor = \operatorname{ord}_p n.$$

Problem B1

Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim\limits_{\theta \to 0} |EF|$. [Here, $|PQ|$ denotes the length of the line segment PQ.]



Solution: The answer is $1/3$.

Put A, B, C into the complex plane with $A = 0$, $C = 1$, and B at positive angle θ. Then $B = 1 + i\tan\theta$ and $D = e^{i\theta}$.

Define $z = (1 - \cos\theta)/(\cos\theta - \cos 2\theta)$ and $s = \sin\theta + z(\sin\theta - \sin 2\theta)$. From the power series $\sin\theta = \theta + \cdots$ and $\cos\theta = 1 - \theta^2/2 + \cdots$ we see that $z = 1/3 + \cdots$, $s = (2/3)\theta + \cdots$, and $\tan\theta = \theta + \cdots$, so $\lim_{\theta\to 0}(s/\tan\theta) = 2/3$. Note that $z > 0$ and $0 < s < \tan\theta$ for sufficiently small θ.

Now define $X = 1 + is$. Then $X - D = z(e^{i\theta} - e^{2i\theta}) = ze^{i\theta}(C - D)$, so $\angle CDX = \theta$; and X is on the line segment from C to B. Hence $E = X$. Consequently $F = s/\tan\theta + is$ and $|E - F| = 1 - s/\tan\theta$, so $\lim_{\theta\to 0}|E - F| = 1 - 2/3 = 1/3$.

Problem B2

Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots. [The roots may be either real or complex.]

Solution: Let r be a complex number. Write $P(x)$ as $p_n(x-r)^n + p_{n-1}(x-r)^{n-1} + \cdots + p_k(x-r)^k$ with p_k nonzero, and write $Q(x)$ as $q_2(x-r)^2 + q_1(x-r) + q_0$.

If $k \geq 2$ then $P''(x) = n(n-1)p_n(x-r)^{n-2} + \cdots + k(k-1)p_k(x-r)^{k-2}$, with $k(k-1)p_k$ nonzero. Compare coefficients of $(x-r)^{k-2}$, $(x-r)^{k-1}$, and $(x-r)^k$ in $P(x)$ and $Q(x)P''(x)$ to see that $q_0 = 0$, $q_1 = 0$, and $q_2 = 1/k(k-1)$; compare coefficients of $(x-r)^n$ to see that $q_2 = 1/n(n-1)$; consequently $k = n$.

In short, if P has a repeated root r, then $P(x) = p_n(x-r)^n$, so P does not have two distinct roots.

Problem B3

Let $A = \{(x, y) : 0 \leq x, y < 1\}$. For $(x, y) \in A$, let

$$S(x, y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{\substack{(x,y)\to(1,1)\\(x,y)\in A}} (1 - xy^2)(1 - x^2y)S(x, y).$$

Solution: All sums here are absolutely convergent.

Define $T_n = \sum_{n/2 \le m \le 2n} x^m$. If n is even then $T_n = (x^{n/2} - x^{2n+1})/(1-x)$; if n is odd then $T_n = (x^{(n+1)/2} - x^{2n+1})/(1-x)$. Thus

$$\begin{aligned}(1-x)S(x,y) = (1-x)\sum_{n\ge 1} T_n y^n &= \sum_{k\ge 1} x^k y^{2k} + \sum_{k\ge 0} x^{k+1}y^{2k+1} - \sum_{n\ge 1} x^{2n+1}y^n \\ &= \frac{xy^2}{1-xy^2} + \frac{xy}{1-xy^2} - \frac{x^3y}{1-x^2y} = \frac{(xy^2+xy)(1-x^2y) - x^3y(1-xy^2)}{(1-x^2y)(1-xy^2)} \\ &= \frac{(1-x)(xy+xy^2+x^2y+x^2y^2-x^3y^3)}{(1-x^2y)(1-xy^2)}.\end{aligned}$$

The desired limit is simply $\lim(xy + xy^2 + x^2y + x^2y^2 - x^3y^3) = 3$.

Problem B4

Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are positive for all x. Suppose that $f'''(x) \le f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Solution: Observe that $\lim_{x\to-\infty} f(x)$ exists and is nonnegative; $\lim_{x\to-\infty} f'(x) = 0$; and $\lim_{x\to-\infty} f''(x) = 0$.

Define $p = 4ff' - 2f''f''$. Then $\lim_{x\to-\infty} p(x) = 0$, and $p' = 4f'f' + 4(f - f''')f'' \ge 4f'f' > 0$, so $p > 0$.

Next define $q = 3ff - 2f'f''$. Then $\lim_{x\to-\infty} q(x) \ge 0$, and $q' = p + 2(f - f''')f' \ge p > 0$, so $q > 0$.

Finally define $r = 3fff - 2f'f'f'$. Then $\lim_{x\to-\infty} r(x) \ge 0$, and $r' = 3f'q > 0$, so $r > 0$. Hence $f'(x)^3 < (3/2)f(x)^3 < (2f(x))^3$.

Problem B5

For an integer $n \ge 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.

Solution: Define $\zeta = e^{i\theta}$ and $v_m = (\zeta^m, \zeta^{2m}, \ldots, \zeta^{nm})$. Note that $v_t \cdot v_m$ is n if $t + m$ is divisible by n, otherwise 0. Consequently $v_0, v_1, \ldots, v_{n-1}$ are linearly independent; the dual basis is $v_0/n, v_{n-1}/n, \ldots, v_1/n$.

Now $Ax = (v_1/2)(v_1 \cdot x) + (v_{n-1}/2)(v_{n-1} \cdot x)$ for any vector x. Indeed, if $x = (x_1, \ldots, x_n)$ then $(2Ax)_j = \sum_k (\zeta^{j+k} + \zeta^{-j-k})x_k = \zeta^j \sum_k \zeta^k x_k + \zeta^{-j} \sum_k \zeta^{-k} x_k = (v_1)_j(v_1 \cdot x) + (v_{n-1})_j(v_{n-1} \cdot x)$.

Hence $Av_1 = (n/2)v_{n-1}$, $Av_{n-1} = (n/2)v_1$, and $Av_m = 0$ if $m = 0$ or if $2 \le m \le n-2$.

Thus $I + A$ is similar to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & n/2 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & n/2 & 0 & \cdots & 1 \end{pmatrix},$$

with determinant $1 - (n/2)^2$.

Problem B6

Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime. [Here, $\gcd(a, b)$ denotes the greatest common divisor of a and b.]

Solution: The following solution is stolen from Thomas Horine.

Define $p(t)$ as the smallest prime dividing t. Define $n = \operatorname{lcm}\{p(t) : t \in S\}$. By hypothesis there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Now $p(s)$ is a common divisor of s and n, so $\gcd(s, n) = s$, i.e., s divides n.

Next define q as the largest prime dividing s. Then q divides n, so $q = p(t)$ for some $t \in S$. Now q divides s exactly once, and divides t; any prime smaller than q does not divide t; any prime larger than q does not divide s. Therefore $\gcd(s, t) = q$.

Comments

A2 is an ancient result, already known to many of the contestants.

In A4 and B3, is one required to prove convergence of the sums? This is straightforward in both cases, but it takes time away from other problems. Contestants should not have to guess how their work will be graded.

In B1, the condition $|AC| = 1$ should have been stated earlier; otherwise it is nonsense to choose D "so that $|AC| = |AD| = 1$." The limit should have been stated for $\theta \to 0^+$. The diagram supplied on the official competition was visibly inaccurate, with $\angle ACB$ larger than a right angle and $\angle BFE$ larger than $\angle BAC$; if the question writers were trying to hide the fact that $\angle ADC = \angle BDE$, they should have left out the diagram entirely.

It isn't clear what B2 means for $n = -\infty$. The polynomial 0 has "at least two distinct roots," and it has at least $-\infty$ distinct roots, but it does not have *exactly* $-\infty$ distinct roots. The writers should have said that n is a nonnegative integer, or said that n is an integer larger than 2, or simply fixed $n = 1999$.

The solution to B5 can be rephrased without complex numbers: the cosine-sine vectors $v_1 + v_{-1}, (v_1 - v_{-1})/i, v_2 + v_{-2}, (v_2 - v_{-2})/i, \ldots$ are independent real eigenvectors of $I + A$ having eigenvalues $1 + n/2, 1 - n/2, 1, 1, \ldots$.

B6 may be rephrased as follows. Let S be a finite collection of nonempty finite sets. Assume that, for every finite set n, S has an element contained in n or disjoint from n. Then there are $s, t \in S$ with $s \cap t$ of size 1. Presumably this result already appears in the combinatorics literature.

—Daniel J. Bernstein, `djb@cr.yp.to`, 6 December 1999