Putnam Mathematical Competition, 2 December 2000

Problem A1

Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, x_2, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Problem A2

Prove that there exist infinitely many integers n such that n, $n+1$, and $n+2$ are each the sum of two squares of integers.

[*Example*: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

Problem A3

The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5 and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

Problem A4

Show that the improper integral

$$\lim_{B\to\infty} \int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

Problem A5

Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

Problem A6

Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

Problem B1

Let a_j, b_j, and c_j be integers for $1 \leq j \leq N$. Assume, for each j, that at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

Problem B2

Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$. [Here $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ and $\gcd(m,n)$ is the greatest common divisior of m and n.]

Problem B3

Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and $a_N \neq 0$. Let N_k denote the number of zeros (including multiplicities) of $\dfrac{d^k f}{dt^k}$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \qquad \text{and} \qquad \lim_{k\to\infty} N_k = 2N.$$

Problem B4

Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

Problem B5

Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows:

Integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n.

Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

Problem B6

Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space, with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Solutions

D. J. Bernstein, 3 December 2000

Problem A1

Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, x_2, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Solution: One can achieve any real number s with $0 < s < A^2$ as follows. Define $u = s/A^2$; then $0 < u < 1$. Define $r = (1-u)/(1+u)$; then $0 < r < 1$. Define $x_j = A(1-r)r^j$; then $x_j > 0$. Finally $\sum x_j = A(1-r)\sum r^j = A$ and $\sum x_j^2 = A^2(1-r)^2 \sum r^{2j} = A^2(1-r)^2/(1-r^2) = A^2(1-r)/(1+r) = A^2 u = s$.

One cannot achieve any other number, since $0 < \sum x_j^2 < (\sum x_j)^2 = A^2$.

Problem A2

Prove that there exist infinitely many integers n such that n, $n+1$, and $n+2$ are each the sum of two squares of integers.

[*Example*: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

Solution: There are infinitely many integers n of the form $2k^2(k+1)^2$; note that $n = (k^2+k)^2 + (k^2+k)^2$, $n+1 = (k^2+2k)^2 + (k^2-1)^2$, and $n+2 = (k^2+k+1)^2 + (k^2+k-1)^2$.

Problem A3

The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5 and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

Solution: The circle circumscribes a square of area 5, so the circle has radius $\sqrt{5/2}$. Hence the rectangle has sides $\sqrt{2}$ and $\sqrt{8}$. Without loss of generality assume that P_2P_4 has length $\sqrt{2}$.

Put P_2, P_4, P_6, P_8 into the complex plane at $\sqrt{2}(1/2+i)$, $\sqrt{2}(-1/2+i)$, $\sqrt{2}(-1/2-i)$, $\sqrt{2}(1/2-i)$. Put P_1 into the complex plane at $\sqrt{5/2}\exp(i\theta)$; then P_3, P_5, P_7 are at $i\sqrt{5/2}\exp(i\theta)$, $-\sqrt{5/2}\exp(i\theta)$, $-i\sqrt{5/2}\exp(i\theta)$.

The triangles $P_8P_1P_2$ and $P_4P_5P_6$ each have area $\sqrt{5}\cos\theta - 1$. The triangles $P_2P_3P_4$ and $P_6P_7P_8$ each have area $\sqrt{5/4}\cos\theta - 1$. Hence the octagon has area $3\sqrt{5}\cos\theta$. The maximum possible area is $3\sqrt{5}$, achieved for $\theta = 0$.

Problem A4

Show that the improper integral

$$\lim_{B\to\infty}\int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

Solution: Rewrite $\sin x \sin x^2$ as $(\cos(x^2-x)-\cos(x^2+x))/2$. In the improper integral $\int_0^\infty \cos(x^2+x)\,dx$ substitute $u = x^2+x$ to obtain $\int_0^\infty 2\cos u\,du/(\sqrt{1+4u}-1)$. The integrand is negative on $(\pi/2, 3\pi/2)$, positive on $(3\pi/2, 5\pi/2)$, etc. The corresponding integrals form an alternating decreasing series since

$$\int_s^{s+\pi}\frac{2\,|\cos u|\;du}{\sqrt{1+4u}-1} > \int_s^{s+\pi}\frac{2\,|\cos u|\;du}{\sqrt{1+4u+4\pi}-1} = \int_{s+\pi}^{s+2\pi}\frac{2\,|\cos v|\;dv}{\sqrt{1+4v}-1}.$$

Thus $\int_0^\infty \cos(x^2+x)\,dx$ converges. Similar comments apply to $\int_0^\infty \cos(x^2-x)\,dx$.

Problem A5

Three distinct points with integer coordinates lie in the plane on a circle of radius $r>0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

Solution: The following solution is stolen from Dave Rusin.

The triangle formed by the points has area $abc/4r$ where a, b, c are the distances between the points. If $a, b, c < r^{1/3}$ then the area is smaller than $1/4$; but the area is at least $1/2$ since the points have integer coordinates.

Problem A6

Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

Solution: The stated conclusion is false, because the word "either" means that exactly one is true. Presumably the intent was to say that $a_1 = 0$ or $a_2 = 0$.

Fact 1: a_{m-1} divides a_1. Proof: a_{m-1} divides $f(a_{m-1}) - f(0) = a_m - a_1 = -a_1$.

Fact 2: a_1 divides a_n if $n \geq 0$. Proof: If $n = 0$ then $a_n = 0$. Otherwise a_1 divides a_{n-1} by induction, so it divides $f(a_{n-1}) - f(0) = a_n - a_1$.

Fact 3: $a_n - a_{n-1}$ divides $a_{n+k} - a_{n+k-1}$ if $n \geq 1$ and $k \geq 0$. Proof: If $k = 0$ then $a_n - a_{n-1} = a_{n+k} - a_{n+k-1}$. Otherwise $a_n - a_{n-1}$ divides $a_{n+k-1} - a_{n+k-2}$ by induction, so it divides $f(a_{n+k-1}) - f(a_{n+k-2}) = a_{n+k} - a_{n+k-1}$.

Fact 4: $a_n - a_{n-1} \in \{-a_1, a_1\}$ if $1 \leq n \leq m$. Proof: Define $k = m - n$. Then $a_n - a_{n-1}$ divides $a_{n+k} - a_{n+k-1} = a_m - a_{m-1} = -a_{m-1}$, which divides a_1; and a_1 divides $a_n - a_{n-1}$.

Fact 5: $a_2 = 0$. Proof: $a_2 - a_1 \in \{-a_1, a_1\}$. Suppose that $a_2 \neq 0$. Then $a_2 = 2a_1$ and $a_1 \neq 0$, so $m \geq 3$. Observe that $a_n = na_1$ for $n \in \{0, 1, 2\}$, but not for $n = m$. Find the smallest $n \geq 3$ for which $a_n \neq na_1$. Then $a_{n-1} = (n-1)a_1$, so $a_n - a_{n-1} \neq a_1$, so $a_n - a_{n-1} = -a_1$, so $a_n = (n-2)a_1 = a_{n-2}$. By induction $a_k \in \{a_{n-1}, a_{n-2}\}$ for all $k \geq n$. In particular $0 = a_m \in \{a_{n-1}, a_{n-2}\}$. Thus $(n-1)a_1 = 0$ or $(n-2)a_1 = 0$. Contradiction.

I would have written this problem as follows: "Define $a_0 = 0$ and $a_{n+1} = f(a_n)$, where f is a polynomial with integer coefficients. Assume that $a_{2000} = 0$. Prove that $a_2 = 0$."

Problem B1

Let a_j, b_j, and c_j be integers for $1 \leq j \leq N$. Assume, for each j, that at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

Solution: Define $f(u, v, w) = \#\{j : (a_j \bmod 2, b_j \bmod 2, c_j \bmod 2) = (u, v, w)\}$. Define $g(r, s, t) = \#\{j : ra_j + sb_j + tc_j \text{ is odd}\}$. Then

$$
\begin{aligned}
g(1,0,0) &= f(1,0,0) + f(1,0,1) + f(1,1,0) + f(1,1,1), \\
g(0,1,0) &= f(0,1,0) + f(0,1,1) + f(1,1,0) + f(1,1,1), \\
g(0,0,1) &= f(0,0,1) + f(0,1,1) + f(1,0,1) + f(1,1,1), \\
g(1,1,0) &= f(1,0,0) + f(0,1,0) + f(1,0,1) + f(0,1,1), \\
g(1,0,1) &= f(1,0,0) + f(0,0,1) + f(1,1,0) + f(0,1,1), \\
g(0,1,1) &= f(0,1,0) + f(0,0,1) + f(1,1,0) + f(1,0,1), \\
g(1,1,1) &= f(1,0,0) + f(0,1,0) + f(0,0,1) + f(1,1,1).
\end{aligned}
$$

Add: $g(1,0,0) + g(0,1,0) + g(0,0,1) + g(1,1,0) + g(1,0,1) + g(0,1,1) + g(1,1,1) = 4f(1,0,0) + 4f(0,1,0) + 4f(0,0,1) + 4f(1,1,0) + 4f(1,0,1) + 4f(0,1,1) + 4f(1,1,1) = 4N$. Thus $g(r, s, t) \geq 4N/7$ for some (r, s, t).

Problem B2

Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$. [Here $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ and $\gcd(m, n)$ is the greatest common divisor of m and n.]

Solution: Presumably "divisior" means "divisor."

Find integers a, b with $\gcd(m, n) = am + bn$. Then $(\gcd(m, n)/n)\binom{n}{m} = a\binom{n-1}{m-1} + b\binom{n}{m}$.

Problem B3

Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and $a_N \neq 0$. Let N_k denote the number of zeros (including multiplicities) of $\dfrac{d^k f}{dt^k}$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \qquad \text{and} \qquad \lim_{k\to\infty} N_k = 2N.$$

Solution: The stated conclusion is false: f has infinitely many roots. Presumably the intent was to say "roots in $[0,1)$." Does anyone proofread the Putnam problems before they are printed?

Say the roots of f in $[0,1)$ are $r_1 < r_2 < \cdots < r_n$ with multiplicities $m_1, m_2, \ldots, m_n$. Then f' has a root at r_i with multiplicity $m_i - 1$ if $m_i \geq 2$; a root in (r_i, r_{i+1}) for $1 \leq i \leq n-1$; a root in $(r_n, 1 + r_1)$; and possibly more roots. Thus there are at least $1 + (n-1) + \sum_i (m_i - 1) = \sum_i m_i$ roots of f' in $[r_1, 1 + r_1)$, hence in $[0,1)$; and there are exactly $\sum_i m_i$ roots of f in $[0,1)$. Thus $N_0 \leq N_1$. By the same argument $N_1 \leq N_2$, $N_2 \leq N_3$, etc.

Find k_0 such that $\sum_{1\leq j<N} (j/N)^k \, |a_j/a_N| < 1/2$ for all $k \geq k_0$. Abbreviate d/dt as D. I will show that $D^k f$ has exactly $2N$ roots in $[0,1)$ for $k \geq k_0$.

Find a real number s with $(D^k \sin)(2\pi Ns) = 1$. Then $(D^k \sin)(2\pi Nt)$ decreases from 1 at s to -1 at $s + 1/2N$, increases to 1 at $s + 2/2N$, etc. By construction

$$\frac{(D^k f)(t)}{(2\pi N)^k a_N} = (D^k \sin)(2\pi Nt) + \sum_{1\leq j<N} \left(\frac{j}{N}\right)^k \frac{a_j}{a_N} (D^k \sin)(2\pi jt),$$

so $(D^k f)(t)$ has the same sign as $a_N (D^k \sin)(2\pi Nt)$ whenever $\left|(D^k \sin)(2\pi Nt)\right| > 1/2$: in particular, at $s, s + 1/2N, s + 2/2N, \ldots$. Therefore $D^k f$ has at least one root in $[s, s + 1/2N)$.

It is not possible for $D^k f$ to have two roots in $[s, s+1/2N)$. Indeed, the roots are in the subinterval $[s + 1/6N, s + 1/3N]$ where $(D^k \sin)(2\pi Nt)$ is in $[-1/2, 1/2]$. If there were two roots then $D^{k+1} f$ would also have a root in the subinterval, so $(D^{k+1} \sin)(2\pi Nt)$ would be in $[-1/2, 1/2]$; contradiction.

The same comments apply to $[s + 1/2N, s + 2/2N)$ and so on. Thus $D^k f$ has exactly $2N$ roots in $[s, s+1)$, hence in $[0,1)$.

Problem B4

Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

Solution: Thanks to Kahan for pointing out the role of cos here. My original solution constructed cos manually.

Define $g(y) = f(\cos 2\pi y)$. Then g is continuous; g is even; g has period 1; and $g(2y) = f(\cos 4\pi y) = f(2(\cos 2\pi y)^2 - 1) = 2(\cos 2\pi y)f(\cos 2\pi y) = 2(\cos 2\pi y)g(y)$.

In particular, $g(1/3) = g(-1/3) = g(2/3) = -g(1/3)$, so $g(1/3) = 0$. Thus $g(n{+}1/3) = 0$ for all integers n. In fact, $g((n{+}1/3)/2^k) = 0$ for all n and all $k \geq 0$. Indeed, if $k \geq 1$, then $g((n{+}1/3)/2^{k-1}) = 0$ by induction, and $\cos(2\pi(n{+}1/3)/2^k) \neq 0$, so $g((n{+}1/3)/2^k) = 0$.

The set $\{(n + 1/3)/2^k\}$ is dense, so g is 0 everywhere. Thus f is 0 on the range of cos, namely $[-1, 1]$.

Robin Chapman comments that one can remove the $2\cos 2\pi y$ factor by considering $f(\cos 2\pi y)/\sin 2\pi y$ for all non-integer y.

Problem B5

Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows:

$$\text{Integer } a \text{ is in } S_{n+1} \text{ if and only if exactly one of } a-1 \text{ or } a \text{ is in } S_n.$$

Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

Solution: Define a polynomial f_n as $\sum_{a \in S_n} x^a$. Then $f_{n+1} \equiv (x+1)f_n \pmod 2$, so $f_n \equiv (x+1)^n f_0$.

In particular, if n is a power of 2 larger than $\deg f_0$, then $f_n \equiv (x+1)^n f_0 \equiv (x^n+1)f_0 = x^n f_0 + f_0$, and all coefficients of $x^n f_0 + f_0$ are 0 or 1, so $f_n = x^n f_0 + f_0$; i.e., $a \in S_n$ if and only if $a \in S_0$ or $a - n \in S_0$.

Problem B6

Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space, with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Solution: The following solution is a composite of solutions from several other people.

Define $A = \{(\pm 1, \pm 1, \ldots, \pm 1)\}$. For each $p \in A$ define $\Delta_p = \{q \in B : |p - q| = 2\}$. Then $\sum_{p \in A} \#\Delta_p = \sum_{q \in B} \#\{p \in A : |p - q| = 2\} = \sum_{q \in B} n = n\#B > 2^{n+1} = 2\#A$. Thus $\#\Delta_p > 2$ for some $p \in A$. Any distinct $q_1, q_2, q_3 \in \Delta_p$ form an equilateral triangle in B.