Putnam Mathematical Competition, 6 December 2003

Problem A1

Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$, there are four ways: 4, $2+2$, $1+1+2$, $1+1+1+1$.

Problem A2

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n}.$$

Problem A3

Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

Problem A4

Suppose that a, b, c, A, B, C are real numbers, $a \neq 0$ and $A \neq 0$, such that

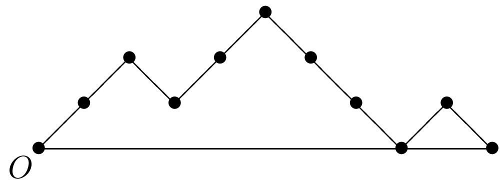
$$\left|ax^2 + bx + c\right| \leq \left|Ax^2 + Bx + C\right|$$

for all real numbers x. Show that

$$\left|b^2 - 4ac\right| \leq \left|B^2 - 4AC\right|.$$

Problem A5

A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

Problem A6

For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

Problem B1

Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Problem B2

Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$, by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries and continue until the final sequence produced consists of a single number x_n. Show that $x_n < \frac{2}{n}$.

Problem B3

Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

Problem B4

Let $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r_4$, then r_1r_2 is a rational number.

Problem B5

Let A, B and C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let a, b, c be the distances from P to A, B, C respectively. Show that there is a triangle with side lengths a, b, c, and that the area of this triangle depends only on the distance from P to O.

Problem B6

Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq \int_0^1 |f(x)| \, dx.$$

Solutions

D. J. Bernstein, 7 December 2003

Problem A1

Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$, there are four ways: 4, $2+2$, $1+1+2$, $1+1+1+1$.

Solution: There are exactly n ways to write n as such a sum. More precisely, there is exactly 1 way $(a_1, a_2, \ldots, a_k)$ for each $k \in \{1, 2, \ldots, n\}$.

Say $a_1, a_2, \ldots, a_k$ satisfy the stated conditions. Observe first that $n \ge a_1 + a_2 + \cdots + a_k \ge 1 + 1 + \cdots + 1 = k$ so $k \in \{1, 2, \ldots, n\}$. The inequalities $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$ imply that all of $a_1, a_2, \ldots, a_k$ are in $\{a_1, a_1 + 1\}$. Define j as the number of occurrences of $a_1 + 1$; then $n = a_1 + a_2 + \cdots + a_k = ka_1 + j$ with $0 \le j \le k - 1$, so $a_1 = \lfloor n/k \rfloor$ and $j = n \bmod k$. Thus $a_1, a_2, \ldots, a_k$ consist of $n \bmod k$ occurrences of $\lfloor n/k \rfloor + 1$ preceded by $k - (n \bmod k)$ occurrences of $\lfloor n/k \rfloor$.

Conversely, take any $k \in \{1, 2, \ldots, n\}$, and build $a_1, a_2, \ldots, a_k$ as $n \bmod k$ occurrences of $\lfloor n/k \rfloor + 1$ preceded by $k - (n \bmod k)$ occurrences of $\lfloor n/k \rfloor$. Then $a_1 \le a_2 \le \cdots \le a_k$; $a_k \le \lfloor n/k \rfloor + 1 \le a_1 + 1$; and $a_1 + a_2 + \cdots + a_k = k \lfloor n/k \rfloor + (n \bmod k) = n$.

Problem A2

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \le ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n}.$$

Solution: If $a_i = b_i = 0$ then the left side and right side are both 0. So assume that $a_i + b_i > 0$ for each i. By the arithmetic-geometric mean inequality,

$$\begin{aligned}
&\left(\frac{a_1}{a_1+b_1} \cdots \frac{a_n}{a_n+b_n}\right)^{1/n} + \left(\frac{b_1}{a_1+b_1} \cdots \frac{b_n}{a_n+b_n}\right)^{1/n} \\
&\quad \le \frac{1}{n}\left(\frac{a_1}{a_1+b_1} + \cdots + \frac{a_n}{a_n+b_n}\right) + \frac{1}{n}\left(\frac{b_1}{a_1+b_1} + \cdots + \frac{b_n}{a_n+b_n}\right) = 1.
\end{aligned}$$

Clear denominators: $(a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n} \le ((a_1 + b_1) \cdots (a_n + b_n))^{1/n}$.

This result, Hölder's inequality, is fairly standard course material, so it isn't a reasonable Putnam problem.

Problem A3

Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

Solution: The problem does not make sense as stated: the trigonometric functions are not all defined when x is a multiple of $\pi/2$. I presume that the intent was to say "for real numbers x where $\sin x \neq 0$ and $\cos x \neq 0$," i.e., for real numbers x that are not multiples of $\pi/2$.

The statement of the problem also implies that there *is* a minimum value of the function. Are contestants required to prove this, or are they allowed to assume it? I presume that contestants are required to prove it.

Anyway, the minimum is $2\sqrt{2} - 1$.

Write $y = \sin x + \cos x$. Then $y^2 = (\sin x)^2 + (\cos x)^2 + 2\sin x \cos x = 1 + 2\sin x \cos x$, so $\tan x + \cot x = (\sin x)^2/(\sin x \cos x) + (\cos x)^2/(\sin x \cos x) = 2/(y^2 - 1)$ and $\sec x + \csc x = (\sin x)/(\sin x \cos x) + (\cos x)/(\sin x \cos x) = 2y/(y^2 - 1)$, so $\sin x + \cos x + \tan x + \cot x + \sec x + \csc x = y + 2/(y^2 - 1) + 2y/(y^2 - 1) = y + 2/(y - 1)$.

If $y > 1$ then, by the arithmetic-geometric-mean inequality, $(y - 1) + 2/(y - 1) \geq 2\sqrt{(y-1)2/(y-1)} = 2\sqrt{2}$, so $y + 2/(y - 1) \geq 2\sqrt{2} + 1 > 2\sqrt{2} - 1$. If $y < 1$ then similarly $(1 - y) + 2/(1 - y) \geq 2\sqrt{2}$ so $-(y + 2/(y - 1)) \geq 2\sqrt{2} - 1$. In both cases, $|y + 2/(y - 1)| \geq 2\sqrt{2} - 1$, so $|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x| \geq 2\sqrt{2} - 1$.

To see that the alleged minimum is achieved, note that $1/\sqrt{2} - 1 \in [-1, 1]$, and set $x = \pi/4 + \arccos(1/\sqrt{2} - 1)$. Then $y = \sqrt{2}\cos(x - \pi/4) = 1 - \sqrt{2}$ so $y + 2/(y - 1) = 1 - 2\sqrt{2}$.

Problem A4

Suppose that a, b, c, A, B, C are real numbers, $a \neq 0$ and $A \neq 0$, such that

$$\left|ax^2 + bx + c\right| \leq \left|Ax^2 + Bx + C\right|$$

for all real numbers x. Show that

$$\left|b^2 - 4ac\right| \leq \left|B^2 - 4AC\right|.$$

Solution: Assume without loss of generality that $a > 0$. (Otherwise replace (a, b, c) with $(-a, -b, -c)$; this transformation does not change $\left|ax^2 + bx + c\right|$, and it does not change $\left|b^2 - 4ac\right|$.) Similarly assume that $A > 0$.

Observe that $\lim_{x\to\infty}(Ax^2+Bx+C)/x^2 = A$. Thus $\lim_{x\to\infty}\left|(Ax^2+Bx+C)/x^2\right| = A$. Similarly $\lim_{x\to\infty}\left|(ax^2+bx+c)/x^2\right| = a$. Hence $a \le A$.

Now write $d = b^2 - 4ac$ and $D = B^2 - 4AC$.

Case 1: $D > 0$. Write $r = (-B+\sqrt{D})/2A$ and $s = (-B-\sqrt{D})/2A$; then $r \ne s$. If $x \in \{r, s\}$ then $Ax^2+Bx+C = 0$ so $\left|ax^2+bx+c\right| \le |0| = 0$ so $ax^2+bx+c = 0$. Thus $ax^2+bx+c = a(x-r)(x-s)$; so $d = a^2(r-s)^2 = a^2(\sqrt{D}/A)^2 = Da^2/A^2$. Hence $0 < d \le D$.

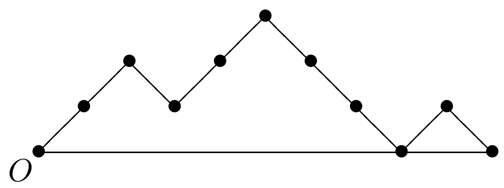
Case 2: $D = 0$. The functions Ax^2+Bx+C and $-Ax^2-Bx-C$ both have value 0 and derivative 0 for $x = -B/2A$, so the intermediate function ax^2+bx+c also has value 0 and derivative 0, i.e., a double root. Hence $d = 0$.

Case 3: $D < 0$. Then $Ax^2+Bx+C = A(x+B/2A)^2 - D/4A > 0$ for all real numbers x, so $\pm(ax^2+bx+c) \le Ax^2+Bx+C$ for all x, so $(A \mp a)x^2 + (B \mp b)x + (C \mp c) \le 0$ for all x, so $(B \mp b)^2 \le 4(A \mp a)(C \mp c)$, i.e., $B^2 \mp 2Bb + b^2 \le 4AC + 4ac \mp (4Ac + 4aC)$. Average $\mp = +$ and $\mp = -$ to see that $B^2 + b^2 \le 4AC + 4ac$, i.e., $d \le -D$.

On the other hand, take $x = -B/2A$ to see that $-d/4a \le a(x+b/2a)^2 - d/4a = ax^2+bx+c \le Ax^2+Bx+C = -D/4A$; i.e., $-d \le -Da/A \le -D$.

Problem A5

A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

Solution: The problem fails to specify the range of n. The definition of a (-1)-path is unclear, so I presume that the definition of n-paths is for all $n \ge 0$ and that the conclusion is for all $n \ge 1$.

Fix $n \ge 1$. If $k \in \{1, 2, \ldots, n\}$ then any $(n-k)$-path, followed by an upstep, followed by any $(k-1)$-path, followed by a downstep, forms an n-path. Every n-path is obtained in this way from a unique k, a unique $(n-k)$-path, and a unique $(k-1)$-path. (The upstep, $(k-1)$-path, and downstep are the last mountain in the n-path.)

Hence $C_n = C_{n-1}C_0 + C_{n-2}C_1 + \cdots + C_0C_{n-1}$ for $n \geq 1$, where C_n is the number of n-paths. Note that $C_0 = 1$.

Next define O_n as the number of n-paths having a final return of odd length. Note that $O_0 = 0$. For $n \geq 1$, every such path is obtained from a unique $k \in \{1, 2, \ldots, n\}$, a unique $(n-k)$-path, and a unique $(k-1)$-path that does not have a final return of odd length; hence $O_n = C_{n-1}(C_0 - O_0) + C_{n-2}(C_1 - O_1) + \cdots + C_0(C_{n-1} - O_{n-1})$.

Next define X_n as the number of n-paths having no return of even length. Note that $X_0 = 1$. For $n \geq 1$, every such path is obtained from a unique $k \in \{1, 2, \ldots, n\}$, a unique $(n-k)$-path having no return of even length, and a unique $(k-1)$-path that does not have a final return of odd length; hence $X_n = X_{n-1}(C_0 - O_0) + X_{n-2}(C_1 - O_1) + \cdots + X_0(C_{n-1} - O_{n-1})$.

In particular, $X_1 = X_0(C_0 - O_0) = 1(1-0) = 1 = C_0$ as desired.

Assume inductively that $X_1 = C_0$, $X_2 = C_1$, and so on through $X_n = C_{n-1}$. Then $X_{n+1} = X_n(C_0 - O_0) + X_{n-1}(C_1 - O_1) + \cdots + X_1(C_{n-1} - O_{n-1}) + X_0(C_n - O_n) = C_{n-1}(C_0 - O_0) + C_{n-2}(C_1 - O_1) + \cdots + C_0(C_{n-1} - O_{n-1}) + (C_n - O_n) = O_n + C_n - O_n = C_n$.

In other words, the number of n-paths with no return of even length is the same as the number of $(n-1)$-paths; i.e., there is a one-to-one correspondence between the two sets.

Problem A6

For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

Solution: Yes.

Define $A = \{0, 3, 5, 6, 9, \ldots\}$ as the set of nonnegative integers n whose binary expansions have an even number of 1's, and define $B = \{1, 2, 4, 7, 8, \ldots\}$ as the set of nonnegative integers n whose binary expansions have an odd number of 1's.

Define f as the formal power series $\sum_{n \in A} x^n$. Similarly define $g = \sum_{n \in B} x^n$.

Now $f(x^2) + xg(x^2) = \sum_{n \in A} x^{2n} + \sum_{n \in B} x^{2n+1} = \sum_{2n \in A} x^{2n} + \sum_{2n+1 \in A} x^{2n+1} = \sum_{m \in A} x^m = f$. Similarly $g(x^2) + xf(x^2) = g$. Hence $f - g = f(x^2) - g(x^2) + xg(x^2) - xf(x^2) = (1-x)(f(x^2) - g(x^2))$.

Furthermore, $f + g = \sum_n x^n = 1/(1-x)$. Hence $f^2 - g^2 = (f-g)(f+g) = f(x^2) - g(x^2)$.

On the other hand, $f^2 = \sum_{s \in A, t \in A} x^s x^t = \sum_{s \in A} x^{2s} + \sum_{s \in A, t \in A, s \neq t} x^{s+t} = f(x^2) + \sum_n r_A(n)x^n$. Similarly $g^2 = g(x^2) + \sum_n r_B(n)x^n$. Hence $\sum_n r_A(n)x^n = f^2 - f(x^2) = g^2 - g(x^2) = \sum_n r_B(n)x^n$; i.e., $r_A(n) = r_B(n)$ for all n.

Problem B1

Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Solution: No.

Suppose that $1+xy+x^2y^2 = a(x)c(y)+b(x)d(y)$. Write $a(x), b(x), c(y), d(y)$ respectively as $a_0 + a_1x + a_2x^2 + \cdots, b_0 + b_1x + b_2x^2 + \cdots, c_0 + c_1y + c_2y^2 + \cdots, d_0 + d_1y + d_2y^2 + \cdots$. Then

$$\begin{aligned}1 + xy + x^2y^2 &= (a_0c_0 + b_0d_0) + (a_0c_1 + b_0d_1)y + (a_0c_2 + b_0d_2)y^2 \\ &\quad + (a_1c_0 + b_1d_0)x + (a_1c_1 + b_1d_1)xy + (a_1c_2 + b_1c_2)xy^2 \\ &\quad + (a_2c_0 + b_2c_0)x^2 + (a_2c_1 + b_2c_1)x^2y + (a_2c_2 + b_2c_2)x^2y^2 + \cdots.\end{aligned}$$

Extract coefficients: $a_0c_0 + b_0d_0 = 1$, so $a_0c_0c_1 + b_0c_1d_0 = c_1$; and $a_0c_1 + b_0d_1 = 0$, so $a_0c_0c_1 + b_0c_0d_1 = 0$, so $b_0(c_1d_0 - c_0d_1) = c_1$. Similarly $b_0(c_1d_2 - d_1c_2) = 0$ and $b_2(c_1d_2 - d_1c_2) = c_1$. If $b_0 = 0$ then $c_1 = 0$; if $b_0 \neq 0$ then $c_1d_2 - d_1c_2 = 0$ so $c_1 = 0$. Either way, $c_1 = 0$. Exchange a, c with b, d to see that $d_1 = 0$. Hence $1 = a_1c_1 + b_1d_1 = 0$; contradiction.

Problem B2

Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$, by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries and continue until the final sequence produced consists of a single number x_n. Show that $x_n < \frac{2}{n}$.

Solution: The kth repeated average of $u_1, u_2, \ldots$ is $(\binom{k}{0}u_1 + \binom{k}{1}u_2 + \cdots + \binom{k}{k}u_k)/2^k$, $(\binom{k}{0}u_2 + \binom{k}{1}u_3 + \cdots + \binom{k}{k}u_{k+1})/2^k$, $\ldots$, by induction on k.

In particular, the $(n-1)$st repeated average of $1, 1/2, \ldots, 1/n$, namely x_n, is

$$\begin{aligned}\frac{1}{2^{n-1}} \sum_{0 \le i \le n-1} \binom{n-1}{i} \frac{1}{i+1} &= \frac{2}{n \cdot 2^n} \sum_{0 \le i \le n-1} \binom{n-1}{i} \frac{n}{i+1} \\ &= \frac{2}{n \cdot 2^n} \sum_{0 \le i \le n-1} \binom{n}{i+1} = \frac{2}{n \cdot 2^n}(2^n - 1) < \frac{2}{n}.\end{aligned}$$

Problem B3

Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

Solution: Let p be a prime number. For any positive integer u, write $\operatorname{ord}_p u$ for the largest nonnegative integer e such that p^e divides u.

If $p^e \leq m < p^{e+1}$ then $\operatorname{ord}_p \operatorname{lcm}\{1, 2, \ldots, m\} = e$. Indeed, p^e appears in $\{1, 2, \ldots, m\}$, so it divides $\operatorname{lcm}\{1, 2, \ldots, m\}$, while p^{e+1} divides none of $\{1, 2, \ldots, m\}$, and hence does not divide $\operatorname{lcm}\{1, 2, \ldots, m\}$.

Hence $\operatorname{ord}_p \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\} = e$ exactly when $p^e \leq \lfloor n/i \rfloor < p^{e+1}$.

Next consider the sum $[i \leq n/p] + [i \leq n/p^2] + [i \leq n/p^3] + \cdots$, where $[\cdots]$ means 1 when $\cdots$ is true, 0 otherwise. This sum is e exactly when $i \leq n/p^e$ but $i > n/p^{e+1}$, i.e., when $p^e \leq \lfloor n/i \rfloor < p^{e+1}$. Hence

$$\begin{aligned}
\operatorname{ord}_p \prod_i \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\} &= \sum_i \operatorname{ord}_p \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\} \\
&= \sum_i ([i \leq n/p] + [i \leq n/p^2] + [i \leq n/p^3] + \cdots) \\
&= \lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots = \operatorname{ord}_p n!.
\end{aligned}$$

This is true for every prime p, so $\prod_i \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\} = n!$.

Problem B4

Let $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

Solution: Abbreviate $r_1 + r_2$ as u, and $r_3 + r_4$ as v. Then

$$\begin{aligned}
f(z)/a &= (z - r_1)(z - r_2)(z - r_3)(z - r_4) = (z^2 - uz + r_1 r_2)(z^2 - vz + r_3 r_4) \\
&= z^4 - (u + v)z^3 + (uv + r_1 r_2 + r_3 r_4)z^2 - (u r_3 r_4 + v r_1 r_2)z + r_1 r_2 r_3 r_4.
\end{aligned}$$

Thus $u + v$ is rational, and by hypothesis u is rational, so v is rational, so uv and $u - v$ are rational. Furthermore, $uv + r_1 r_2 + r_3 r_4$ is rational, so $r_1 r_2 + r_3 r_4$ is rational, so $u r_1 r_2 + u r_3 r_4$ is rational. Next, $u r_3 r_4 + v r_1 r_2$ is rational, so $(u - v) r_1 r_2$ is rational. By hypothesis $u - v$ is nonzero, so $r_1 r_2$ is rational.

This is too easy for a B4 problem.

Problem B5

Let A, B and C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let a, b, c be the distances from P to A, B, C respectively. Show that there is a triangle with side lengths a, b, c, and that the area of this triangle depends only on the distance from P to O.

Solution: Put the circle into the complex plane, translated so that $O = 0$ and rotated so that $A = 1$. Then B and C are the two primitive cube roots of 1.

Define $\Delta = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$. Heron's theorem states that a, b, c are the side lengths of a triangle if and only if $\Delta > 0$, and that the triangle has area $\sqrt{\Delta}/4$.

Write P as ru where r is a nonnegative real number and u is a complex number of absolute value 1. Then $a^2 = |P - A|^2 = |ru - 1|^2 = (ru - 1)(r/u - 1) = s - ru - r/u$ where $s = 1 + r^2$; $b^2 = |P - B|^2 = |ru - B|^2 = (ru - B)(r/u - C) = s - Cru - Br/u$; and $c^2 = |P - C|^2 = |ru - C|^2 = (ru - C)(r/u - B) = s - Bru - Cr/u$.

Square to obtain a^4, b^4, c^4, and add. All of the u's drop out of the resulting formula, since $1 + B + C = 0$ and $1 + B^2 + C^2 = 0$. (For example, the cross-terms $2rsu$, $2Crsu$, and $2Brsu$ add up to 0.) Hence $a^4 + b^4 + c^4 = 3(s^2 + 2r^2)$. By a similar computation, $a^2b^2 + a^2c^2 + b^2c^2 = 3s^2 + (2B + 2C + B^2 + C^2)r^2 = 3(s^2 - r^2)$. Hence $\Delta = 3(s^2 - 4r^2) = 3(1 - r^2)^2$.

By hypothesis P is in the interior of the unit circle; i.e., $0 \le r < 1$. Hence $\Delta > 0$. The area of the triangle is $(1 - r^2)\sqrt{3}/4$, which is determined by $r = |P|$ as claimed.

Problem B6

Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \ge \int_0^1 |f(x)| \, dx.$$

Solution: If $x_1, \ldots, x_n$ are real numbers in $[0, 1]$ then

$$\sum_{i \ne j} 2[f(x_i)f(x_j) < 0] \min\{|f(x_i)|, |f(x_j)|\} \le n \sum_j |f(x_j)|$$

by Lemma 1 below. Integrate over all $x_1, \ldots, x_n$:

$$n(n-1) \int \int 2[f(x)f(y) < 0] \min\{|f(x)|, |f(y)|\} \, dx \, dy \le n^2 \int |f(x)| \, dx.$$

Divide by n^2:

$$\frac{n-1}{n}\int\int 2[f(x)f(y)<0]\min\{|f(x)|,|f(y)|\}\,dx\,dy \le \int |f(x)|\,dx.$$

Take the limit as $n\to\infty$:

$$\int\int 2[f(x)f(y)<0]\min\{|f(x)|,|f(y)|\}\,dx\,dy \le \int |f(x)|\,dx.$$

Finally, $|u+v| = |u|+|v| - 2[uv<0]\min\{|u|,|v|\}$, so

$$\begin{aligned}&\int\int |f(x)+f(y)|\,dx\,dy\\ &\quad= \int |f(x)|\,dx + \int |f(y)|\,dy - 2\int\int [f(x)f(y)<0]\min\{|f(x)|,|f(y)|\}\,dx\,dy\\ &\quad\ge \int |f(y)|\,dy\end{aligned}$$

as claimed.

Lemma 1: $\sum_{i\ne j} 2[r_ir_j<0]\min\{|r_i|,|r_j|\} \le n\sum_j |r_j|$ for any real numbers $r_1, r_2, \ldots, r_n$.

Proof: Without loss of generality assume that $|r_1| \ge |r_2| \ge \cdots \ge |r_n|$. Then the sum on the left is equal to $\sum_{i<j} 4[r_ir_j<0]\,|r_j|$.

Define a_j for $j\ge 0$ as the number of positive entries in $r_1, r_2, \ldots, r_j$. Define b_j for $j\ge 0$ as the number of negative entries in $r_1, r_2, \ldots, r_j$. The product a_jb_j is at most $(j/2)^2 \le (n/2)(j/2)$, so

$$\begin{aligned}&(a_1b_1-a_0b_0)\,|r_1| + (a_2b_2-a_1b_1)\,|r_2| + \cdots + (a_nb_n-a_{n-1}b_{n-1})\,|r_n|\\ &= a_1b_1(|r_1|-|r_2|) + a_2b_2(|r_2|-|r_3|) + \cdots + a_{n-1}b_{n-1}(|r_{n-1}|-|r_n|) + a_nb_n(|r_n|)\\ &\le \frac{n}{2}\frac{1}{2}(|r_1|-|r_2|) + \frac{n}{2}\frac{2}{2}(|r_2|-|r_3|) + \cdots + \frac{n}{2}\frac{n-1}{2}(|r_{n-1}|-|r_n|) + \frac{n}{2}\frac{n}{2}(|r_n|)\\ &= \frac{n}{4}\,|r_1| + \frac{n}{4}\,|r_2| + \frac{n}{4}\,|r_3| + \cdots + \frac{n}{4}\,|r_n|\,.\end{aligned}$$

Next observe that $a_jb_j - a_{j-1}b_{j-1}$ is exactly b_{j-1} if $r_j>0$; a_{j-1} if $r_j<0$; and 0 if $r_j=0$. In other words, $a_jb_j-a_{j-1}b_{j-1}$ is the number of $i<j$ such that $r_ir_j<0$. Hence $\sum_{i<j} 4[r_ir_j<0]\,|r_j| = \sum_j 4(a_jb_j-a_{j-1}b_{j-1})\,|r_j| \le n\sum_j |r_j|$ as claimed.