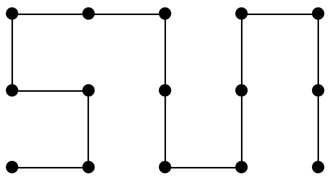
Putnam Mathematical Competition, 3 December 2005

Problem A1

Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.
(For example, $23 = 9 + 8 + 6$.)

Problem A2

Let $S = \{(a, b) \mid a = 1, 2, \ldots, n,\ b = 1, 2, 3\}$. A *rook tour* of S is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that (i) $p_i \in S$, (ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$, (iii) for each $p \in S$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?
(An example of such a rook tour for $n = 5$ is depicted below.)



Problem A3

Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

Problem A4

Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Problem A5

Evaluate $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1} dx$.

Problem A6

Let n be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are the P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

Problem B1

Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor v \rfloor$ is the greatest integer less than or equal to v.)

Problem B2

Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Problem B3

Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

Problem B4

For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \le m$. Show that $f(m, n) = f(n, m)$.

Problem B5

Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

(a) $$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) P(x_1, \ldots, x_n) = 0 \qquad \text{(identically)}$$

and that

(b) $$x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).$$

Show that $P = 0$ identically.

Problem B6

Let S_n denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $v(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{v(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1}.$$

Solutions

D. J. Bernstein, 4 December 2005

Problem A1

Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.
(For example, $23 = 9 + 8 + 6$.)

Solution: For each $n \ge 0$ define a sequence $E(n)$ of elements of $2^{\mathbf{N}}3^{\mathbf{N}}$ as follows:

- if $n = 0$ then $E(n)$ is the empty sequence ();
- if $n > 0$ and n is even then $E(n)$ is $2E(n/2)$, the sequence obtained by doubling each component of $E(n/2)$;
- if $n > 0$ and n is odd then $E(n)$ is $(E(n-3^k), 3^k)$, the sequence obtained by appending 3^k to $E(n-3^k)$, where k is the largest integer such that $3^k \le n$.

I claim that the sum of $E(n)$ is n; that each component of $E(n)$ is even if n is even; and that no component of $E(n)$ divides another component. Proof:

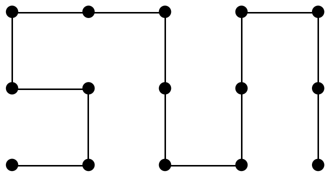
- $n = 0$: $E(n)$ is empty so it has sum 0.
- $n > 0$ and n is even: Assume inductively that $E(n/2)$ has sum $n/2$ and that no component of $E(n/2)$ divides another component. Then $E(n) = 2E(n/2)$ has sum $2(n/2) = n$; each component of $E(n)$ is even; and no component divides another component.
- $n > 0$ and n is odd: Find the largest integer k such that $3^k \le n$. Note that $n - 3^k$ is even. Assume inductively that $E(n-3^k)$ has sum $n-3^k$; that each component of $E(n-3^k)$ is even; and that no component of $E(n-3^k)$ divides another component. Then $E(n) = (E(n-3^k), 3^k)$ has sum $(n-3^k)+3^k = n$; each component of $E(n-3^k)$, being even, does not divide 3^k; and each component of $E(n-3^k)/2$, being at most $(n-3^k)/2 < (3^{k+1}-3^k)/2 = 3^k$, is not divisible by 3^k, so each component of $E(n-3^k)$ is not divisible by 3^k.

In particular, for $n \ge 1$, the components of $E(n)$ are one or more elements of $2^{\mathbf{N}}3^{\mathbf{N}}$, adding up to n, none dividing the others.

Problem A2

Let $S = \{(a,b) \mid a = 1,2,\ldots,n,\ b = 1,2,3\}$. A *rook tour* of S is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that (*i*) $p_i \in S$, (*ii*) p_i and p_{i+1} are a unit distance apart, for $1 \le i < 3n$, (*iii*) for each $p \in S$ there is a

unique i such that $p_i = p$. How many rook tours are there that begin at $(1,1)$ and end at $(n,1)$?
(An example of such a rook tour for $n = 5$ is depicted below.)



Solution: The answer is 0 for $n = 1$ and 2^{n-2} for $n \geq 2$.

For each $n \geq 1$, define r_n as the number of $n \times 3$ rook tours beginning at $(1,1)$ and ending at $(n,1)$, and define s_n as the number of $n \times 3$ rook tours beginning at $(1,1)$ and ending at $(n,3)$.

One can obtain an $n \times 3$ rook tour beginning at $(1,1)$ and ending at $(n,1)$ as follows. Choose $k \in \{1, 2, \ldots, n-1\}$. Take a $k \times 3$ rook tour beginning at $(1,1)$ and ending at $(k,3)$. Step to the right $n-k$ times to $(n,3)$; then down once to $(n,2)$; then left $n-k-1$ times to $(k+1,2)$; then down once to $(k+1,1)$; then right $n-k-1$ times to $(n,1)$.

Every $n \times 3$ rook tour beginning at $(1,1)$ and ending at $(n,1)$ can be obtained uniquely in this way. Indeed, the final step down on the tour must be from $(k+1,2)$ to $(k+1,1)$ for a unique $k \in \{1, 2, \ldots, n-1\}$; it must be followed by $n-k-1$ steps right to $(n,1)$; it must be preceded by $n-k-1$ steps left from $(n,2)$, since any earlier step up (or left) would prevent the tour from reaching $(n,2)$; $(n,2)$ must be preceded by a step down from $(n,1)$; and $(n,1)$ must be preceded by $n-k$ steps right from $(k,1)$.

Consequently $r_n = s_1 + s_2 + \cdots + s_{n-1}$ for all $n \geq 1$. In particular, $r_1 = 0$.

Similarly, one can obtain an $n \times 3$ rook tour beginning at $(1,1)$ and ending at $(n,3)$ by choosing $k \in \{1, 2, \ldots, n-1\}$, taking a $k \times 3$ rook tour beginning at $(1,1)$ and ending at $(k,1)$, stepping to the right $n-k$ times, stepping up once, stepping left $n-k-1$ times, stepping up once, and stepping right $n-k-1$ times; *or* by simply starting from $(1,1)$, stepping to the right $n-1$ times, stepping up once, stepping left $n-1$ times, stepping up once, and stepping right $n-1$ times. Every $n \times 3$ rook tour beginning at $(1,1)$ and ending at $(n,3)$ can be obtained uniquely in this way.

Consequently $s_n = r_1 + r_2 + \cdots + r_{n-1} + 1$ for all $n \geq 1$. In particular, $s_1 = 1$.

Now $r_n = s_n = 2^{n-2}$ for all $n \geq 2$. Indeed, assume inductively that $r_k = s_k = 2^{k-2}$ for $2 \leq k < n$. Then $r_n = s_1 + s_2 + \cdots + s_{n-1} = 1 + 2^0 + \cdots + 2^{n-3} = 2^{n-2}$ and $s_n = r_1 + r_2 + \cdots + r_{n-1} + 1 = 0 + 2^0 + \cdots + 2^{n-3} + 1 = 2^{n-2}$.

Problem A3

Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

Solution: I'm annoyed by this problem, for two reasons.

First, the definition of $g(z)$ is ambiguous when n is odd. Does $z^{n/2}$ mean the principal branch of the $n/2$ power, applied to z? Or does it mean the principal branch of the square root, applied to z^n? Or is z not actually a complex number, but an element of a Riemann sheet chosen so that the square root does not need a branch cut? My proof works for any of these choices of g, but I can imagine proofs that work with all the roots of g and that occasionally break down for the first two choices of g. The problem should have said "Show that all zeros of $zp'(z) - (n/2)p(z)$ have absolute value 1."

Second, the statement is false for $n = 0$. Consider, for example, $p(z) = 1$. This is a polynomial of degree 0, and it has no zeros, so all of its zeros have absolute value 1. The function $g(z) = p(z)/z^{n/2}$ is then constant, so its derivative is 0, so its derivative has every complex number (in the ambiguous domain) as a root, not just complex numbers of absolute value 1.

Assume from now on that $n \geq 1$. Factor $p(z)$ as $p_n(z - r_1)(z - r_2) \cdots (z - r_n)$. By hypothesis each r_i has absolute value 1. If $|z| > 1$ then, by Lemma 2, $(z + r_j)/(z - r_j)$ has positive real part for each j, so—since $n \geq 1$—$\sum_j (z + r_j)/(z - r_j)$ has positive real part. Similarly, if $|z| < 1$ then $\sum_j (z + r_j)/(z - r_j)$ has negative real part. Either way $\sum_j (z + r_j)/(z - r_j)$ is nonzero; i.e., $\sum_j 2z/(z - r_j) \neq \sum_j (z - r_j)/(z - r_j) = n$; i.e., $2zp'(z)/p(z) \neq n$; i.e., $g'(z) \neq 0$.

Lemma 1: If $z \in \mathbf{C}$ then $(z+1)/(z-1)$ has positive real part when $|z| > 1$ and negative real part when $|z| < 1$.

Proof: Write z in polar coordinates as $re^{i\theta}$. Then $(z+1)/(z-1) = (re^{i\theta} + 1)/(re^{i\theta} - 1)$ has real part $(r^2 - 1)/((r\cos\theta - 1)^2 + (r\sin\theta)^2)$, which is positive if $r > 1$ and negative if $0 \leq r < 1$.

Lemma 2: If $r \in \mathbf{C}$, $|r| = 1$, and $|z| \in \mathbf{C}$, then $(z+r)/(z-r)$ has positive real part when $|z| > 1$ and negative real part when $|z| < 1$.

Proof: Apply Lemma 1 to z/r.

Problem A4

Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Solution: Write $v_1, v_2, \ldots, v_b$ for the length-n row vectors covering H. By assumption each entry of v_i is ± 1, so v_i has squared length n. By assumption $v_1, v_2, \ldots, v_b$ are pairwise orthogonal, so $v_1 + v_2 + \cdots + v_b$ has squared length nb. On the other hand, by assumption $v_1 + v_2 + \cdots + v_b$ has a entries equal to b, so $v_1 + v_2 + \cdots + v_b$ has squared length at least ab^2. Thus $ab^2 \le nb$; consequently $ab \le n$, whether or not $b = 0$.

Problem A5

Evaluate $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1} dx$.

Solution: The answer is $(\pi/8)\log 2$. Fast proof by Bhargava, Kedlaya, and Ng: The integral is $\int_0^{\pi/4} \log(\tan\theta + 1)\, d\theta = \int_0^{\pi/4}((1/2)\log 2 + \log\cos(\pi/4 - \theta) - \log\cos\theta)\, d\theta = \int_0^{\pi/4}((1/2)\log 2)\, d\theta = (\pi/8)\log 2$ since $\int_0^{\pi/4} \log\cos(\pi/4 - \theta)\, d\theta = \int_0^{\pi/4} \log\cos\theta\, d\theta$.

Problem A6

Let n be given, $n \ge 4$, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are the P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

Solution: The answer is $(n^2 - 2n)/2^{n-1}$.

Define $\theta(p, q) \in [0, 2\pi)$, where p and q are points on the circle, as the angle *from p to q* in the clockwise direction.

Define a vertex as “happy” if it is immediately *before* an acute-angled vertex. In other words, if v, w, x are consecutive vertices in clockwise order, then v is happy if and only if the angle at vertex w is acute. Equivalently: v is happy if and only if $\theta(v, x) > \pi$. Equivalently: v is happy if and only if at most one other vertex P has $\theta(v, P) \le \pi$.

Critical fact 1: P_1 is happy with probability $n/2^{n-1}$.

Indeed, here is a partition of P_1’s happiness into n disjoint possibilities, each occurring with probability $1/2^{n-1}$:

- $\theta(P_1, P_i) > \pi$ for all $i \ne 1$;
- $\theta(P_1, P_i) > \pi$ for all $i \notin \{1, 2\}$ while $\theta(P_1, P_2) \le \pi$;
- $\theta(P_1, P_i) > \pi$ for all $i \notin \{1, 3\}$ while $\theta(P_1, P_3) \le \pi$;
- $\ldots$;
- $\theta(P_1, P_i) > \pi$ for all $i \notin \{1, n\}$ while $\theta(P_1, P_n) \le \pi$.

Critical fact 2: P_1 and P_2 are simultaneously happy with probability $1/2^{n-3}(n-1)$.

This probability is the average, over P_1, P_2, of the conditional probability given P_1, P_2. I claim that the conditional probability is exactly $((\pi-\alpha)^{n-2}+(n-2)\alpha(\pi-\alpha)^{n-3})/(2\pi)^{n-2}$ where $\alpha = \min\{\theta(P_1,P_2), \theta(P_2,P_1)\}$. The distribution of α is uniform over $[0,\pi]$, so the average of $((\pi-\alpha)^{n-2}+(n-2)\alpha(\pi-\alpha)^{n-3})/(2\pi)^{n-2}$ is

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\pi \frac{(\pi-\alpha)^{n-2}+(n-2)\alpha(\pi-\alpha)^{n-3}}{(2\pi)^{n-2}}d\alpha \\
&\quad= \frac{1}{\pi}\int_0^\pi \frac{(3-n)(\pi-\alpha)^{n-2}+(n-2)\pi(\pi-\alpha)^{n-3}}{(2\pi)^{n-2}}d\alpha \\
&\quad= \frac{1}{2^{n-2}\pi^{n-1}}\left((3-n)\frac{\pi^{n-1}}{n-1}+(n-2)\pi\frac{\pi^{n-2}}{n-2}\right) \\
&\quad= \frac{3-n}{2^{n-2}(n-1)}+\frac{n-1}{2^{n-2}(n-1)} = \frac{1}{2^{n-3}(n-1)}.
\end{aligned}$$

Proof of the claim: Assume without loss of generality that $\theta(P_1,P_2) \le \pi$, i.e., that $\alpha = \theta(P_1,P_2)$. Now P_1 is happy if and only if $\theta(P_1,P_3),\ldots,\theta(P_1,P_n)$ are all $> \pi$; and P_2 is happy if and only if at most one of $\theta(P_2,P_3),\ldots,\theta(P_2,P_n)$ is $\le \pi$. Thus P_1 and P_2 are simultaneously happy if and only if one of the following disjoint events occurs:

- $\pi+\alpha < \theta(P_1,P_i) < 2\pi$ for each $i \notin \{1,2\}$—which, given P_1 and P_2, occurs with conditional probability $(\pi-\alpha)^{n-2}/(2\pi)^{n-2}$;
- $\pi+\alpha < \theta(P_1,P_i) < 2\pi$ for each $i \notin \{1,2,3\}$ while $\pi < \theta(P_1,P_3) \le \pi+\alpha$—which, given P_1 and P_2, occurs with conditional probability $\alpha(\pi-\alpha)^{n-3}/(2\pi)^{n-2}$;
- $\pi+\alpha < \theta(P_1,P_i) < 2\pi$ for each $i \notin \{1,2,4\}$ while $\pi < \theta(P_1,P_4) \le \pi+\alpha$—which, given P_1 and P_2, occurs with conditional probability $\alpha(\pi-\alpha)^{n-3}/(2\pi)^{n-2}$;
- $\ldots$;
- $\pi+\alpha < \theta(P_1,P_i) < 2\pi$ for each $i \notin \{1,2,n\}$ while $\pi < \theta(P_1,P_n) \le \pi+\alpha$—which, given P_1 and P_2, occurs with conditional probability $\alpha(\pi-\alpha)^{n-3}/(2\pi)^{n-2}$.

Add to obtain $((\pi-\alpha)^{n-2}+(n-2)\alpha(\pi-\alpha)^{n-3})/(2\pi)^{n-2}$ as claimed.

Critical fact 3: P_1, P_2, P_3 are all happy with probability 0.

Indeed, in each of the above ways for P_1, P_2 to be happy, P_3 is visibly unhappy: either $\pi+\alpha < \theta(P_1,P_3) \le 2\pi$, in which case both $\theta(P_3,P_1)$ and $\theta(P_3,P_2)$ are below π, or $\pi < \theta(P_1,P_3) \le \pi+\alpha$ while $\pi+\alpha < \theta(P_1,P_4) \le 2\pi$, in which case both $\theta(P_3,P_1)$ and $\theta(P_3,P_4)$ are below π. This is where the proof uses the hypothesis that $n \ge 4$.

Putting it all together: Permute indices to see that P_i is happy with probability $n/2^{n-1}$; that P_i, P_j are simultaneously happy with probability $1/2^{n-3}(n-1)$, if the indices i, j are distinct; and that P_i, P_j, P_k are simultaneously happy with probability 0, if i, j, k are distinct. By inclusion-exclusion, the probability of at least one happy vertex is $n(n/2^{n-1}) - \binom{n}{2}(1/2^{n-3}(n-1)) = (n^2-2n)/2^{n-1}$.

Problem B1

Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor v \rfloor$ is the greatest integer less than or equal to v.)

Solution: One answer is the nonzero polynomial $P(x, y) = (y - 2x)(y - 2x - 1)$.

Define $i = \lfloor a \rfloor$. Then $i \le a < i + 1$. If $i \le a < i + 0.5$ then $2i \le 2a < 2i + 1$ so $\lfloor 2a \rfloor = 2i = 2\lfloor a \rfloor$ so $\lfloor 2a \rfloor - 2\lfloor a \rfloor = 0$. Otherwise $i + 0.5 \le a < i + 1$ so $2i + 1 \le 2a < 2i + 2$ so $\lfloor 2a \rfloor = 2i + 1 = 2\lfloor a \rfloor + 1$ so $\lfloor 2a \rfloor - 2\lfloor a \rfloor - 1 = 0$. Either way $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$.

Problem B2

Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Solution: $1, 1$; $3, 2, 3, 6$; $3, 2, 6, 3$; $3, 3, 2, 6$; $3, 3, 6, 2$; $3, 6, 2, 3$; $3, 6, 3, 2$; $4, 4, 4, 4, 4$.

By inspection each of these possibilities works. Conversely, assume that $k_1 + \cdots + k_n = 5n - 4$ and $1/k_1 + \cdots + 1/k_n = 1$; I will show that $n, k_1, \ldots, k_n$ is one of these possibilities.

If $k_1, \ldots, k_n$ are all equal then $1 = 1/k_1 + \cdots + 1/k_n = n/k_1$ so $k_1 = n$ and $5n - 4 = k_1 + \cdots + k_n = nk_1 = n^2$. Hence $(n - 4)(n - 1) = n^2 - 5n + 4 = 0$. Either $n = 1$, in which case $(n, k_1, \ldots, k_n) = (1, 1)$; or $n = 4$, in which case $(n, k_1, \ldots, k_n) = (4, 4, 4, 4, 4)$.

Assume from now on that $k_1, \ldots, k_n$ are not all equal. The average of $k_1, \ldots, k_n$ is $(5n - 4)/n$ so the geometric average of $k_1, \ldots, k_n$ is below $(5n - 4)/n$. The average of $1/k_1, \ldots, 1/k_n$ is $1/n$ so the geometric average of $1/k_1, \ldots, 1/k_n$ is below $1/n$. Thus the geometric average of $k_1, \ldots, k_n, 1/k_1, \ldots, 1/k_n$ is below $\sqrt{((5n - 4)/n)(1/n)}$; but this geometric average is equal to 1. Therefore $1 < (5n - 4)/n^2$; so $(n - 1)(n - 4) < 0$; so $1 < n < 4$; so $n = 2$ or $n = 3$.

If $n = 2$ then $k_1 + k_2 = 5n - 4 = 6$ so $1/k_1 + 1/k_2$ is one of $1/1 + 1/5$, $1/2 + 1/4$, $1/3 + 1/3$, none of which equal 1.

If $n = 3$ then $k_1 + k_2 + k_3 = 5n - 4 = 11$ so $1/k_1 + 1/k_2 + 1/k_3$ is one of $1/1 + \cdots, 1/2 + 1/2 + \cdots, 1/2 + 1/3 + 1/6, 1/2 + 1/4 + 1/5, 1/3 + 1/3 + 1/5, 1/3 + 1/4 + 1/4$. By inspection none of these are 1 except $1/2 + 1/3 + 1/6$. Thus (k_1, k_2, k_3) is a permutation of $(2, 3, 6)$.

Problem B3

Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

Solution: Here are two classes of qualifying functions f:

- Define $f(x) = x$. Then $f'(x) = 1$ so $f'(1/x) = 1 = x/f(x)$.
- Choose positive real numbers α, β with $\beta \neq 1$, and define $f(x) = \alpha x^\beta$. Then $f'(a/x)f(x) = \alpha\beta(a/x)^{\beta-1}\alpha x^\beta = \alpha^2\beta a^{\beta-1}x = x$ where $a = (1/\alpha^2\beta)^{1/(\beta-1)}$.

I claim that there are no other possibilities: if $f'(a/x) = x/f(x)$ then f is one of the above functions. Indeed, substitute a/x for x: $f'(x) = a/xf(a/x)$. The right side is differentiable, so the left side is too:

$$f''(x) = \frac{-a}{(xf(a/x))^2}\left(xf'(a/x)\frac{-a}{x^2} + f(a/x)\right).$$

Substitute $f(a/x) = a/xf'(x)$ and $f'(a/x) = x/f(x)$:

$$\begin{aligned} f''(x) &= \frac{-a}{(a/f'(x))^2}\left(\frac{x^2}{f(x)}\frac{-a}{x^2} + \frac{a}{xf'(x)}\right) \\ &= \frac{-f'(x)^2}{a}\left(\frac{-a}{f(x)} + \frac{a}{xf'(x)}\right) = \frac{f'(x)^2}{f(x)} - \frac{f'(x)}{x}. \end{aligned}$$

Define $g(x) = \log f(x)$. Then $g'(x) = f'(x)/f(x)$; note that $f'(x) > 0$ so $g'(x) > 0$. Differentiate again:

$$g''(x) = \frac{f(x)f''(x) - f'(x)^2}{f(x)^2} = \frac{-f(x)f'(x)/x}{f(x)^2} = \frac{-f'(x)}{xf(x)} = \frac{-g'(x)}{x}.$$

Define $h(x) = \log g'(x)$. Then $h'(x) = g''(x)/g'(x) = -1/x$. Integrate: there is a real number d such that $h(x) = d - \log x$. Exponentiate: $g'(x) = \beta/x$ where $\beta = \exp d$. Integrate again: there is a real number c such that $g(x) = c + \beta\log x$. Exponentiate: $f(x) = \alpha x^\beta$ where $\alpha = \exp c$. If $\beta = 1$ then $f(x) = \alpha x$ so $\alpha = f'(a/x) = x/f(x) = 1/\alpha$ so $\alpha = 1$ so $f(x) = x$ as claimed. Otherwise α, β are positive real numbers, $\beta \neq 1$, and $f(x) = \alpha x^\beta$ as claimed.

Problem B4

For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

Solution: Extend the same definition to all nonnegative integers m, n.

If $n = 0$ then there is exactly one n-tuple, and its sum of absolute values is $0 \leq m$. Thus $f(m, 0) = 1$.

If $m = 0$ then the only qualifying n-tuple is $(0, 0, \ldots, 0)$. Thus $f(0, n) = 1$.

If $n \geq 1$ and $m \geq 0$ then one can construct a qualifying n-tuple as follows: choose x_n in $\{-m, -m+1, \ldots, m-1, m\}$; choose an $(n-1)$-tuple $(x_1, x_2, \ldots, x_{n-1})$ satisfying $|x_1| + |x_2| + \cdots + |x_{n-1}| \leq m - |x_n|$. Every qualifying n-tuple arises uniquely in this way. Thus $f(m, n) = f(m, n-1) + 2f(m-1, n-1) + 2f(m-2, n-1) + \cdots + 2f(0, n-1)$.

Consequently, $f(m+1, n+1) = f(m, n+1) + f(m+1, n) + f(m, n)$ if $n \geq 0$ and $m \geq 0$. Indeed, $f(m, n+1) = f(m, n) + 2f(m-1, n) + 2f(m-2, n) + \cdots + 2f(0, n)$. and $f(m+1, n+1) = f(m+1, n) + 2f(m, n) + 2f(m-1, n) + 2f(m-2, n) + \cdots + 2f(0, n)$; subtract.

Theorem: $f(m, n) = f(n, m)$ for all nonnegative integers m, n. Proof: If $m = 0$ then $f(m, n) = f(0, n) = 1 = f(n, 0) = f(n, m)$ as claimed. If $n = 0$ then $f(m, n) = f(m, 0) = 1 = f(0, m) = f(n, m)$ as claimed. So assume that $m \geq 1$ and $n \geq 1$. Then $f(m, n) = f(m-1, n) + f(m, n-1) + f(m-1, n-1)$ and $f(n, m) = f(n-1, m) + f(n, m-1) + f(n-1, m-1)$. Induct on $m + n$.

Alternate approaches: One can, with marginally more work, prove the symmetric formula $f(m, n) = 1 + 2(m+n-1)!/(m-1)!(n-1)!$. One can use other bijections; partitioning by choices of x_n is straightforward but might not produce the shortest proof.

Problem B5

Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

$$\text{(a)} \qquad \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) P(x_1, \ldots, x_n) = 0 \qquad \text{(identically)}$$

and that

$$\text{(b)} \qquad x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).$$

Show that $P = 0$ identically.

Solution: Assume that $n \geq 1$. Define $X = x_1^2 + \cdots + x_n^2$ and $D = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. The problem is to show that if X divides P and $D(P) = 0$ then $P = 0$.

Suppose that $P \neq 0$. Find the maximum positive integer e such that X^e divides P. Write P/X^e as $\sum_{i \geq 0} H_i$ where H_i is homogeneous of degree i.

Then $0 = D(P) = D(\sum_i X^e H_i) = \sum_i D(X^e H_i)$. The terms $D(X^e H_i)$ are homogeneous of different degrees, namely $2e - 2 + i$, so $D(X^e H_i) = 0$ for each i. Thus $XD(H_i) + e(4(e-1) + 4 \deg H_i + 2n)H_i = 0$ by Lemma 2. The coefficient $e(4(e-1) + 4 \deg H_i + 2n)$ is positive since $n \geq 1$ and $e \geq 1$; thus H_i is a multiple of X. This is true for every i, so P/X^e is a multiple of X, contradicting the definition of e.

Lemma 1: If H is homogeneous then $D(XH) = XD(H) + (4\deg H + 2n)H$.

Proof: $\dfrac{\partial^2 XH}{\partial x_i^2} = X\dfrac{\partial^2 H}{\partial x_i^2} + 2\dfrac{\partial X}{\partial x_i}\dfrac{\partial H}{\partial x_i} + H\dfrac{\partial^2 X}{\partial x_i^2} = X\dfrac{\partial^2 H}{\partial x_i^2} + 4x_i\dfrac{\partial H}{\partial x_i} + 2H$. By homogeneity $\sum_i x_i(\partial H/\partial x_i) = (\deg H)H$.

Lemma 2: If H is homogeneous and $e \geq 0$ then

$$D(X^e H) = X^e D(H) + e(4(e-1) + 4\deg H + 2n)X^{e-1}H.$$

Proof: For $e = 0$: $D(X^e H) = D(H) = X^e D(H) + e(\cdots)$. For $e \geq 1$: $D(X^e H) = XD(X^{e-1}H) + (4\deg X^{e-1}H + 2n)X^{e-1}H$ by Lemma 1. Assume inductively that $D(X^{e-1}H) = X^{e-1}D(H) + (e-1)(4(e-2) + 4\deg H + 2n)X^{e-2}H$. Then

$$\begin{aligned}
D(X^e H) &= X^e D(H) + (e-1)(4(e-2) + 4\deg H + 2n)X^{e-1}H \\
&\quad + (4\deg X^{e-1}H + 2n)X^{e-1}H \\
&= X^e D(H) + (4(e-1)(e-2) + 4(e-1)\deg H + 2(e-1)n \\
&\quad + 4(e-1)\deg X + 4\deg H + 2n)X^{e-1}H \\
&= X^e D(H) + (4(e-1)(e) + 4e\deg H + 2en)X^{e-1}H
\end{aligned}$$

since $\deg X = 2$.

Problem B6

Let S_n denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $v(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{v(\pi) + 1} = (-1)^{n+1}\frac{n}{n+1}.$$

Solution: Define e_n as the number of even permutations of $\{1, 2, \ldots, n\}$. Recall that $e_n = 1$ if $n = 0$; $e_n = 1$ if $n = 1$; and $e_n = n!/2$ if $n \geq 2$.

Define f_k as the number of even derangements of $\{1, 2, \ldots, k\}$, i.e., the number of even permutations with no fixed points. Define g_k as the number of odd derangements of $\{1, 2, \ldots, k\}$, i.e., the number of odd permutations with no fixed points.

By choosing k elements of $\{1, \ldots, n\}$, choosing an even derangement of those k elements, and fixing the other $n - k$ elements, one obtains an even permutation of $\{1, \ldots, n\}$ with exactly $n - k$ fixed points. Every such permutation arises in this way. Thus there are exactly $\binom{n}{k} f_k$ even permutations of $\{1, \ldots, n\}$ with exactly $n - k$ fixed points. Sum over k to see that $\sum_{0 \leq k \leq n} \binom{n}{k} f_k = e_n$.

Similarly, there are exactly $\binom{n}{k} g_k$ odd permutations of $\{1, \ldots, n\}$ with exactly $n-k$ fixed points, and $\sum_{0 \le k \le n} \binom{n}{k} g_k = n! - e_n$.

I claim that $f_n - g_n = (-1)^{n-1}(n-1)$ for all $n \ge 0$. Proof: The point is that $f_n - g_n$ is determined recursively by the equation $\sum_k \binom{n}{k}(f_k - g_k) = 2e_n - n!$; so one simply has to check that $\sum_k \binom{n}{k}(-1)^{k-1}(k-1) = 2e_n - n!$. For $n = 0$ the latter sum is $(-1)^{-1}(-1) = 1 = 2e_0 - 0!$ as desired. For $n = 1$ the sum is $(-1)^{-1}(-1) + (-1)^0(0) = 1 = 2e_1 - 1!$ as desired. For $n \ge 2$ one has $\sum_k \binom{n}{k}(-1)^k = (1-1)^n = 0$ and $\sum_k \binom{n}{k}(-1)^{k-1}k = \sum_k \binom{n-1}{k-1}(-1)^{k-1} = (1-1)^{n-1} = 0$ so $\sum_k \binom{n}{k}(-1)^{k-1}(k-1) = 0 = 2e_n - n!$ as desired.

Now if $n \ge 1$ then $\sum_{0 \le k \le n} \binom{n+1}{k}(f_k - g_k) = (-1)^{n+1}n$. Proof: $\binom{n+1}{n+1}(f_{n+1} - g_{n+1}) = (-1)^n n$ and $\sum_{0 \le k \le n+1} \binom{n+1}{k}(f_k - g_k) = 2e_{n+1} - (n+1)! = 0$.

The problem asks for the sum of $\sigma(\pi)/(1 + v(\pi))$ over all permutations π of $\{1, \ldots, n\}$. There are $\binom{n}{k} f_k$ even permutations π with $v(\pi) = n-k$, contributing $\binom{n}{k} f_k/(1+n-k) = \binom{n+1}{k} f_k/(n+1)$ to the sum. There are also $\binom{n}{k} g_k$ odd permutations π with $v(\pi) = n-k$, contributing $-\binom{n}{k} g_k/(1+n-k) = -\binom{n+1}{k} g_k/(n+1)$ to the sum. Overall the sum is $\sum_{0 \le k \le n} \binom{n+1}{k}(f_k - g_k)/(n+1) = (-1)^{n+1}n/(n+1)$ if $n \ge 1$.

Beware that this formula is wrong for $n = 0$. The problem should have said that n is a positive integer.