“Mathematics is like checkers
in being suitable for the young,
not too difficult, amusing,
and without peril to the state.”

—Plato

Counting rational points by brute force

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Elkies, 1988,
"On $A^4 + B^4 + C^4 = D^4$":

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$

. . . "seems beyond the range of reasonable exhaustive computer search."

All solutions ≤ 21000000 with positive coordinates, mod scaling, permutations:

$95800^4 + 217519^4 + 414560^4 = 422481^4$

$673865^4 + 1390400^4 + 2767624^4 = 2813001^4$

$1705575^4 + 5507880^4 + 8332208^4 = 8707481^4$

$5870000^4 + 8282543^4 + 11289040^4 = 12197457^4$

$4479031^4 + 12552200^4 + 14173720^4 = 16003017^4$

$3642840^4 + 7028600^4 + 16281009^4 = 16430513^4$

$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$

(422481 Frye; 2813001 MacLeod; 20615673 Elkies; others new)

Standard method

To find all solutions $\leq H$:

Sort $\left\{(a^4 + b^4, a, b) : a, b \leq H\right\}$
into increasing order
in the first component.
Also $\left\{(d^4 - c^4, c, d) : c, d \leq H\right\}$.

Merge the sorted lists,
looking for collisions.

MSD radix sort takes linear time in realistic machine model.

Time: $H^{2+o(1)}$.
Tolerable for large H.

Space: $H^{2+o(1)}$.
Impossible for large H.

Standard improvements

1. Reduce $\#\{(a, b)\}$, $\#\{(c, d)\}$ by carefully choosing representatives for $\{(a, b, c, d)\}$ mod scaling et al.

2. Chop $\mathbf{Z}$ into intervals in $\mathbf{R}$ or $\mathbf{Q}_p$. Enumerate $a^4 + b^4$ and $d^4 - c^4$ in each interval separately.

3. Prove theorems to exclude solutions in some intervals.

Assume $a\mathbf{Z}+b\mathbf{Z}+c\mathbf{Z}+d\mathbf{Z}=\mathbf{Z}$.

Permute a, b, c so that
$a \in 2\mathbf{Z}$ and $b \in 10\mathbf{Z}$.

Then $a \in 8\mathbf{Z}$, $b \in 40\mathbf{Z}$,
$d-1 \in 8\mathbf{Z}$, $d \notin 5\mathbf{Z}$,
and $c \equiv \pm d \pmod{1024}$.

$\#\{(c,d)\} \approx 10^{-4} H^2$.
Can reduce further with
more p-adic restrictions.

(Morgan Ward, 1948)

Searching without sorting

Factor each $d^4 - c^4$ into primes, write as sum of two squares in all possible ways; check for fourth powers.

No solutions for $H = 10^4$. (Ward)

Time $H^{2+o(1)}$ with modern factoring methods, but still rather slow.

Alternative: For each (c, d),
enumerate possible b's,
see if $d^4 - c^4 - b^4$ is fourth power.

No solutions for $H = 2.2 \cdot 10^5$.
(Lander-Parkin-Selfridge, 1967)

Solutions for $H = 2 \cdot 10^6$.
$\approx 2 \cdot 10^{-6} H^3$ fourth-power tests.
(Frye, 1988)

Sorting without storing

For fixed b, easy to generate $a^4 + b^4$ in increasing order, using very little space.

Run one generator for each b, merge results.

(Lander-Parkin, 1967)

$\underline{2}=1^4+1^4$	$32=2^4+2^4$	$162=3^4+3^4$
$\underline{17}=2^4+1^4$	$32=2^4+2^4$	$162=3^4+3^4$
$82=3^4+1^4$	$\underline{32}=2^4+2^4$	$162=3^4+3^4$
$\underline{82}=3^4+1^4$	$97=3^4+2^4$	$162=3^4+3^4$
$257=4^4+1^4$	$\underline{97}=3^4+2^4$	$162=3^4+3^4$
$257=4^4+1^4$	$272=4^4+2^4$	$\underline{162}=3^4+3^4$
$\underline{257}=4^4+1^4$	$272=4^4+2^4$	$337=4^4+3^4$
$626=5^4+1^4$	$\underline{272}=4^4+2^4$	$337=4^4+3^4$
$626=5^4+1^4$	$641=5^4+2^4$	$\underline{337}=4^4+3^4$
$\underline{626}=5^4+1^4$	$641=5^4+2^4$	$706=5^4+3^4$

. . .

For each of the H^2 outputs,
search for smallest of H results.
Total time: $H^{3+o(1)}$.

Space: H generators.

Heaps

A **heap** is a sequence
$x_1, x_2, \ldots, x_n$ such that
$x_1 \le x_2$, $x_1 \le x_3$,
$x_2 \le x_4$, $x_2 \le x_5$,
$x_3 \le x_6$, $x_3 \le x_7$,
$x_4 \le x_8$, $x_4 \le x_9$,
etc.

e.g. $1, 4, 1, 5, 9, 2, 6, 5$

Smallest element of a heap $x_1, x_2, \ldots, x_n$ is x_1.

For any y, easy to permute $y, x_2, x_3, \ldots, x_n$ into a new heap:

1. $j \leftarrow 1$.
2. $k \leftarrow 2j$.
3. Stop if $k > n$.
4. $k \leftarrow k+1$ if $k < n$, $x_{k+1} < x_k$.
5. Stop if $y \le x_k$.
6. Swap y (in jth spot) with x_k.
7. $j \leftarrow k$.
8. Go back to step 2.

Use heap in Lander-Parkin method.
Space: H generators.
Time: $H^{2+o(1)}$.

Other data structures allowing fast find-and-replace-smallest: leftist trees, loser selection trees, balanced trees, B-trees, etc.

Heaps are small and very fast.

History

Heaps: J. W. J. Williams, 1964.
Improvements: Floyd, 1964.

Using heaps to enumerate sums in sorted order: W. S. Brown. See exercise in Knuth on multiplying sparse power series.

Speeding up Lander-Parkin: Randy Ekl (balanced trees); independently me (heaps); independently David W. Wilson (heaps).

Limiting precision

Search for solutions to
$(a^4 \bmod m) + (b^4 \bmod m) - \delta m$
$= (d^4 \bmod m) - (c^4 \bmod m)$
with $m = 2^{60} - 93$
and $\delta \in \{0, 1, 2\}$.

Use sorted table of
fourth powers mod m.

Other computations

Enumerating rational points on various cubic surfaces.

Distribution seems consistent with best available conjecture.

91 can be written in 2 ways as sum of two coprime cubes: $91 = (-5)^3 + 6^3 = 3^3 + 4^3$.

3367 in 3 ways.

16776487 in 4 ways. (Rathbun)

5064336773593 in 5 ways.
137904678696613339 in 5 ways.

http://pobox.com/~djb
/sortedsums.html

http://pobox.com/~djb
/papers/sortedsums.dvi